Power Round

CHMMC 2018

December 2nd, 2018

1 Euclidean Algorithm (25 pts)

Definition 1.1 (Greatest Common Divisor)**.** The greatest common divisor of two positive integers a and b, denoted $\gcd(a,b)$, is defined to be the greatest positive integer d such that $d \mid a$ and $d \mid b$.

Remark 1.2. The definition of divisibility is $d \mid a$ if and only if there exists an integer q such that $a = qd$.

/2 pts **Problem 1.1.** Prove that if a and b are positive integers such that $a > b$, then $\gcd(a,b) = \gcd(a-b,b)$.

Solution 1. If $d \mid a$ and $d \mid b$, then there exists integer q_a and q_b such that $a = dq_a$ and $b = dq_b$. Therefore $a - b = dq_a - dq_b = d(q_a - q_b)$ and $d \mid a - b$. Similarly, if $d' \mid a - b$ and $d' \mid b$, then $d' \mid a$. Hence any common divisor of a and b is also a common divisor of $a-b$ and b and vice versa, therefore $\gcd(a,b) = \gcd(a-b,b)$.

/4 pts **Problem 1.2.** Prove that if a and b are positive integers such that $a = bq + r$ where $0 \le r < b$, then $\gcd(a,b) = \gcd(b,r)$.

Solution 2. If $d \mid a$ and $d \mid b$, then there exist integers q_a and q_b such that $a = dq_a$ and $b = dq_b$. Therefore $a - bq = d(q_a - qq_b)$ and $d \mid a - bq = r$. Similarly, if $d \mid b$ and $d \mid r$ then there exist integers q_b and q_r such that $b = dq_b$ and $r = dq_r$. Thus $bq + r = d(dq_b + q_r)$ and $d \mid bq + r = a$. Therefore any common divisor of a and b is also a common divisor of b and r and vice versa. Since the set of common divisors are the same, the *greatest* common divisor must also be.

Remark 1.3 (Division Algorithm). For two positive integers a, b, there exists a *unique* quotient and remainder q and r such that $a = bq + r$ where $0 \le r < b$.

/3 pts **Problem 1.3** (Euclidean Algorithm)**.** To calculate the greatest common divisor of two positive integers a and b, we repeatedly apply the division algorithm to obtain a sequence of quotients $q_1, q_2, \ldots$ and remainders $r_1, r_2, \ldots$ such that

$$\begin{aligned} a &= bq_1 + r_1, \quad 0 \le r_1 < b \\ b &= r_1 q_2 + r_2, \quad 0 \le r_2 < r_1 \\ r_1 &= r_2 q_3 + r_3, \quad 0 \le r_3 < r_2 \end{aligned}$$

and so on, for $k \geq 3$,

$$r_{k-2} = r_{k-1}q_k + r_k, \quad 0 \leq r_k < r_{k-1}.$$

Prove this process terminates after finitely many steps, at which point the remainder is zero, that is, $r_{n-1} = r_n q_{n+1}$ for some n. Prove that $r_n = \gcd(a, b)$.

Solution 3. The remainders decrease at each step and are non-negative, therefore, they must reach zero. Applying Problem 1.2 to Equations 1, 2 and 3,

$$\gcd(a, b) = \gcd(b, r_1) = \gcd(r_1, r_2) = \gcd(r_2, r_3).$$

In general, $\gcd(r_{k-2}, r_{k-1}) = \gcd(r_{k-1}, r_k)$ for $3 \leq k \leq n$. At the final step,

$$\gcd(a, b) = \gcd(r_2, r_3) = \cdots = \gcd(r_{k-2}, r_{k-1}) = \gcd(r_{n-1}, r_n) = r_n.$$

/3 pts **Problem 1.4.** Compute $\gcd(100631, 423041)$ using the Euclidean Algorithm.

Solution 4. By the Euclidean Algorithm,

$$\begin{aligned} 423041 &= 100631 \cdot 4 + 20517 \\ 100631 &= 20517 \cdot 4 + 18563 \\ 20517 &= 18563 \cdot 1 + 1954 \\ 18563 &= 1954 \cdot 9 + 977 \\ 1954 &= 977 \cdot 2. \end{aligned}$$

Therefore $\gcd(100631, 423041) = \boxed{977}$.

Definition 1.4 (Game of Euclid)**.** Two players A and B play the following game, where players alternate taking turns, with A going first. The game begins with two positive integers $a > b$. In a turn, a player replaces the larger number by subtracting from it a multiple of the smaller number, such that the result is nonnegative. Play continues until one of the numbers remaining is zero, then the last player to take a turn wins.

Remark 1.5. The description of this game is from a paper by Cole and Davie.

/3 pts **Problem 1.5.** Determine the player with the winning strategy in the Game of Euclid for $(a, b) = (162, 100)$ and $(a, b) = (161, 100)$.

Solution 5. A wins the first game, while B wins the second:

$$(162, 100) \overset{A}{\to} (100, 62) \overset{B}{\to} (62, 38) \overset{A}{\to} (38, 24) \overset{B}{\to} (24, 14) \overset{A}{\to} (14, 10) \overset{B}{\to} (10, 4) \overset{A}{\to} (6, 4) \overset{B}{\to} (4, 2) \overset{A}{\to} (2, 0)$$

$$(161, 100) \overset{A}{\to} (100, 61) \overset{B}{\to} (61, 39) \overset{A}{\to} (39, 22) \overset{B}{\to} (22, 17) \overset{A}{\to} (17, 5) \overset{B}{\to} (7, 5) \overset{A}{\to} (5, 2) \overset{B}{\to} (3, 2) \overset{A}{\to} (2, 1) \overset{B}{\to} (1, 0).$$

Definition 1.6 (Golden Ratio)**.** The two roots of the quadratic $x^2 - x - 1 = 0$ are $\varphi = (1 + \sqrt{5})/2 \approx 1.618$ and $\psi = (1 - \sqrt{5})/2$.

Remark 1.7. Note $162/100 = 1.62 > \varphi > 1.61 = 161/100$.

/5 pts **Problem 1.6.** Prove that if $1 < a/b < \varphi$, there is only one possible move $(a, b) \to (b, a')$, and this satisfies $b/a' > \varphi$.

Solution 6. Since $b < a < b\varphi < 2b$, the only possible move is $(a, b) \to (b, a - b)$. Hence $a' = a - b$. Furthermore, $\varphi^2 - \varphi = 1$, so

$$\frac{b}{a'} = \frac{b}{a-b} = \frac{1}{a/b - 1} > \frac{1}{\varphi - 1} = \varphi.$$

/5 pts **Problem 1.7.** Prove that player A may force a win if $a/b = 1$ or $a/b > \varphi$.

Solution 7. When $\varphi < n/m < 2$, player A moves to $(m, n - m)$ since

$$\frac{m}{n-m} = \frac{1}{n/m - 1} < \frac{1}{\varphi - 1} = \varphi.$$

When $n/m > 2$ and $n \equiv r \pmod{m}$ for $0 \le r < m$, there are two moves:

$$(n, m) \to (m, r) \text{ or } (n, m) \to (m + r, m).$$

If $r = 0$, player A wins. Otherwise, φ is between m/r and $(m+r)/m$. Player A moves to the position whose ratio lies strictly between 1 and φ. Player B is left in position (a, b) where $1 < a/b < \varphi$. Player B must then move to (b, a') where $b/a' > \varphi$, from which the process is repeated and A may force a win.

2 Fibonacci Numbers (38 pts)

Definition 2.1. The Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. For instance, $F_3 = 2, F_4 = 3, F_5 = 5$, and so forth.

/5 pts **Problem 2.1.** Let f_n be the number of ways to tile a board of size $n \times 1$ with squares (size one) and dominoes (size two). Prove $f_n = F_{n+1}$.

Figure 1: Some tilings of a board of size four

Solution 8. If a board of size n begins with a square, then we have to tile a board of size $n - 1$. However, if the board begins with a domino, then we have to tile a board of size $n - 2$. Therefore, $f_n = f_{n-1} + f_{n-2}$. Since $f_1 = F_2 = 1$ and $f_2 = F_3 = 2$, the Fibonacci numbers are shifted by an index so $f_n = F_{n+1}$.

/9 pts **Problem 2.2.** Prove the following Fibonacci identities:

- $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$.
- $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$.
- $F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}$.

Solution 9. For each problem, there are two valid methods: induction or tiling.

- Consider the number of tilings of a board of size $n+1$. Of these, $f_{n+1}-1$ use at least one domino. We consider the position of the final domino, that is, the location $1 \le k \le n$ such that there is a domino covering cells k and $k+1$ and all squares beyond that point. We therefore simply have to tile the first $k-1$ squares of the board, which can be done in f_{k-1} ways. Therefore, $F_{n+2}-1 = f_{n+1}-1 = \sum_{k=1}^{n} f_{k-1} = \sum_{k=1}^{n} F_k = F_1 + \cdots + F_n$.
- Consider f_{2n-1}, the number of tilings of a board of size $2n-1$. By parity, there must be at least one square, therefore, we consider the position of the final square at an odd location $2k-1$ for $1 \le k \le n$. Since there is a square covering cell $2k-1$ and all dominos beyond this point, we simply have to tile the first $2k-2$ squares, which is in f_{2k-2} ways. Therefore, $f_{2n-1} = \sum_{k=1}^{n} f_{2k-2}$ so $F_{2n} = \sum_{k=1}^{n} F_{2k-1} = F_1 + F_3 + \cdots + F_{2n-1}$.
- We proceed by induction. For $n=1$, this is simply $F_1^2 = F_1F_2$. Assume this statement holds for $n=k$, therefore $F_1^2+F_2^2+F_3^2+\cdots+F_k^2 = F_kF_{k+1}$. Then adding the next squared Fibonacci term F_{k+1}^2 to both sides gives

$$\begin{aligned}\left(F_1^2+F_2^2+F_3^2+\cdots+F_k^2\right)+F_{k+1}^2 &= F_kF_{k+1}+F_{k+1}^2\\ &= F_{k+1}\left(F_k+F_{k+1}\right)\\ &= F_{k+1}F_{k+2}.\end{aligned}$$

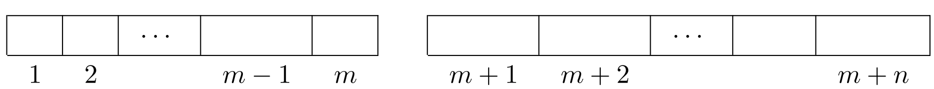
This is the identity for $n=k+1$, therefore by induction it holds for all n.

/6 pts **Problem 2.3.** Prove $F_{a+b} = F_{a+1}F_b + F_aF_{b-1}$ for $a, b \ge 1$.

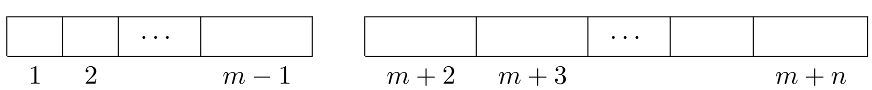
Solution 10. We prove the analogous identity $f_{m+n} = f_mf_n + f_{m-1}f_{n-1}$ first. The left-hand side is simply the number of tilings of a board of size $m+n$.

For the right-hand side, we have two cases:

- If there is no domino at cell m, we have f_mf_n tilings:



- If there is a domino at cells m and $m+1$, we have $f_{m-1}f_{n-1}$ tilings:



Therefore $f_{m+n} = f_mf_n + f_{m-1}f_{n-1}$. Substituting $m=a$ and $n=b-1$,

$$F_{a+b} = f_{a+b-1} = f_af_{b-1} + f_{a-1}f_{b-2} = F_{a+1}F_b + F_aF_{b-1}. \qquad \square$$

Definition 2.2 (Fibonacci Nim)**.** Let there be n coins in a pile and A, B be two players who alternate removing coins from the pile, with A going first. On the first move, a player is not allowed to take all of the coins, and on each subsequent move, the number of coins removed can be at most twice that of the previous move. The winner is the player who removes the final coin(s).

/2 pt **Problem 2.4.** Demonstrate which player has the winning strategy in Fibonacci Nim for $n = 7, 10$.

Solution 11. Notice B's move at each turn is forced: $7 \stackrel{A}{\to} 5 \stackrel{B}{\to} 4 \stackrel{A}{\to} 3 \stackrel{B}{\to} 2 \stackrel{A}{\to} 0$. Similarly for $n = 10$, A plays $10 \stackrel{A}{\to} 8$. If B plays to 7, as seen above, A wins. If B plays $8 \stackrel{B}{\to} 6$, then A responds $6 \stackrel{A}{\to} 5$, from which B is forced to play $5 \stackrel{B}{\to} 4$ and as above, A wins. Therefore A has a winning strategy for $n = 7$ and $n = 10$.

/8 pts **Problem 2.5** (Zeckendorf's Theorem)**.** Prove that every positive integer N can be represented uniquely as a sum of distinct non-consecutive Fibonacci numbers F_k with $k \geq 2$.

Solution 12. For the base case, $1 = F_2$. Now, assume that every every integer up to K can be written uniquely as the sum of non-consecutive Fibonacci numbers. Let $F_{\max}$ be the largest Fibonacci number such that $F_{\max} \leq K + 1$. If $F_{\max} = K + 1$, then we are clearly done. Otherwise, $F_{\max} < K + 1 < F_{\max+1}$, therefore

$$0 < (K + 1) - F_{\max} < F_{\max+1} - F_{\max} = F_{\max-1}. \qquad (\star)$$

By our hypothesis, there exists a sequence $\{a_j\}_{j=1}^{m}$ with $a_{j+1} > a_j + 1$ such that

$$K + 1 - F_{\max} = F_{a_1} + F_{a_2} + \cdots + F_{a_m}.$$

Since $F_{a_m} < F_{\max-1}$ by $(\star)$, adding $F_{\max}$ to both sides produces a valid representation for $K + 1$. The method used here is known as a **greedy strategy**.

To prove uniqueness, we require the following lemma:

Lemma 2.3. *The sum of any set of distinct, non-consecutive Fibonacci numbers whose largest member is F_j is strictly less than the next Fibonacci number F_{j+1}.*

Proof: The next largest member of our set is at most F_{j-2}, whose sum is strictly less than F_{j-1}, therefore the sum of the set is less than $F_{j-1} + F_j = F_{j+1}$.

Assume for the sake of contradiction we have two distinct representations:

$$K + 1 = F_{a_1} + F_{a_2} + \cdots + F_{a_m} = F_{b_1} + F_{b_2} + \cdots + F_{b_l}.$$

Without loss of generality, $a_m \geq b_l$. If $a_m > b_l$, by our Lemma,

$$\begin{aligned} K + 1 = F_{b_1} + F_{b_2} + \cdots + F_{b_l} &< F_{b_l+1} - 1 \\ &\leq F_{a_m} - 1 \\ &< F_{a_1} + F_{a_2} + \cdots + F_{a_m} \\ &= K + 1. \end{aligned}$$

Contradiction. Hence $a_m = b_l$. By our hypothesis, $K + 1 - F_{a_m} = K + 1 - F_{b_l}$ has a unique representation, therefore $K + 1$ also has a unique representation.

Definition 2.4. Consider more general positions in Fibonacci Nim as pairs (q, r) consisting of a number of coins r remaining together with a "quota" q, specifying the maximum number of coins a player may take in the next move. Say that a position is *nice* if q is at least the smallest term in the Zeckendorf representation of r.

/8 pts **Problem 2.6.** (i) Show that given a nice position, there is a move such that the resulting position is not nice. (ii) Show that any move from a position which is not nice results in a nice position. (iii) Determine with proof the starting values of n for which the first player has no winning strategy.

Solution 13. (*Written by Gideon Leeper*)

(i) First, an inductive argument shows that $F_{k+2} > 2F_k$ for all $k \geq 2$: this holds when $k = 2, 3$, and if $F_{k+2} > 2F_k$ and $F_{k+3} > 2F_{k+1}$, then $F_{k+4} = F_{k+3} + F_{k+2} > 2F_{k+1} + 2F_k = F_{k+2}$. Now let (q, r) be a nice position, so writing $r = F_{a_1} + F_{a_2} + \cdots + F_{a_k}$ in its Zeckendorf representation, these terms form an increasing sequence of non-consecutive Fibonacci numbers with all $a_i \geq 2$, and $q \geq F_{a_1}$. Thus the move given by taking F_{a_1} coins is valid, and the resulting position is $(2F_{a_1}, r - F_{a_1})$. Note $r - F_{a_1} = F_{a_2} + F_{a_3} + \cdots + F_{a_k}$, which has least term F_{a_2}, but since we must have $a_2 \geq a_1 + 2$, it follows that $F_{a_2} \geq F_{a_1+2} > 2F_{a_1}$, meaning the smallest term in the Zeckendorf representation of $r - F_{a_1}$ is greater than the quota $2F_{a_1}$, meaning the new position is not nice.

(ii) Let (q, r) be a position which is not nice, so each term in the Zeckendorf representation of r is greater than q. Then any move consists of taking some $0 < x \leq q$ coins, resulting in the position $(2x, r - x)$. Suppose such a position is not nice. Then each term of the Zeckendorf representation of $r - x$ is at least $2x$. This means the smallest term F_a of the representation of $r - x$ is greater than $2x$, while the largest term F_b of the Zeckendorf representation of x is at most x, hence $a \geq b + 2$. Thus concatenating the Zeckendorf representations of $r - x$ and x gives the Zeckendorf representation of r, and since this contains the terms of the representation of x, some term is less than $x \leq q$, which contradicts that (q, r) is not nice. Thus the result of any move from (q, r) is nice.

(iii) First we prove nice positions are winning for the current player. We proceed by strong induction on r. When $r = 1 = F_2$, the current player can take 1 coin and win. If the current player has a winning strategy at nice positions (q, r) for $r = 1, 2, \ldots, k$, then when (q, r) is nice for $r = k+1$, we have two cases. Either $q \geq r$, in which case the current player can take all r coins and win, or $r > q > F_{a_1}$, the smallest term in the Zeckendorf representation of r. In this case, the current player, say A, can take F_{a_1} coins, so the remaining position is not nice by (i). Thus the other player cannot win in one move, and by (ii) any move by B results in a nice position (q', r') with $r' < r$. By the inductive hypothesis, since this position (q', r') is winning, A has a winning strategy to proceed. It also follows that any position which is not nice is a losing position, since any move by the next player A results in a nice position, i.e. a winning position for the other player B. Finally, consider the initial position in which the games starts, namely $(n - 1, n)$. This is not nice if the smallest term of the Zeckendorf representation of n is greater than $n - 1$, or n is a Fibonacci number.

3 Divisibility Sequences (27 pts)

Definition 3.1. A *divisibility sequence* is an integer sequence a_n for $n \geq 1$,

$$m \mid n \Rightarrow a_m \mid a_n.$$

/4 pts **Problem 3.1.** Prove the Fibonacci numbers are a divisibility sequence.

Solution 14. We claim $F_m \mid F_{mq}$ for all natural q. We use proof by induction. For $q = 1$, $F_m \mid F_m$. For $q = 2$, $F_{2m} = F_{m+1}F_m + F_mF_{m-1}$ so $F_m \mid F_{2m}$.

Suppose the statement is true for $q = k$, and we show it holds for $q = k+1$. From Problem 2.3 with $a = mk$ and $b = m$, $F_{mk+m} = F_{mk+1}F_m + F_{mk}F_{m-1}$. Since $F_m \mid F_{mk}$ (hypothesis), $F_m \mid F_{mk+m}$ by the linear combination theorem.

/3 pts **Problem 3.2.** Prove the sequence $a_n = A^n - B^n$ is divisibility for $A > B > 0$.

Solution 15. For a natural number q,

$$\begin{aligned} a_{nq} &= A^{nq} - B^{nq} \\ &= (A^n)^q - (B^n)^q \\ &= (A^n - B^n)\left(A^{n\cdot(q-1)} + A^{n\cdot(q-2)}B^n + \cdots + B^{n\cdot(q-1)}\right). \end{aligned}$$

Therefore $a_n \mid a_{nq}$ implying a_n is a divisibility sequence.

Definition 3.2. A divisibility sequence has *strong divisibility* if for all m, n positive integers, $\gcd(a_m, a_n) = a_{\gcd(m,n)}$.

/3 pts **Problem 3.3.** Prove the sequence $a_n = kn$ for natural k has strong divisibility.

Solution 16. We distribute: $\gcd(a_m, a_n) = \gcd(km, kn) = k\gcd(m, n) = a_{\gcd(m,n)}$.

/8 pts **Problem 3.4.** Prove $a_n = k^n - 1$ for natural k has strong divisibility. (*Hint:* There exist integers x and y such that $\gcd(m, n) = mx + ny$ by Bézout's.)

Solution 17. Let $d = \gcd(k^m - 1, k^n - 1)$ and $k^{\gcd(m,n)} - 1 = a_{\gcd(m,n)}$. Then since $d \mid k^m - 1$, $k^m \equiv 1 \pmod{d}$. Similarly, $k^n \equiv 1 \pmod{d}$. Therefore,

$$k^{\gcd(m,n)} - 1 = k^{mx+ny} - 1 = k^{mx}k^{ny} - 1 \equiv 0 \pmod{d},$$

so $d \mid k^{\gcd(m,n)} - 1$. Similarly, $\gcd(m,n) \mid m$ and $\gcd(m,n) \mid n$, so by Problem 3.2,

$$k^{\gcd(m,n)} - 1 \mid k^m - 1 \text{ and } k^{\gcd(m,n)} - 1 \mid k^n - 1.$$

Thus $k^{\gcd(m,n)} - 1 \mid \gcd(k^m - 1, k^n - 1) = d$ and we conclude $d = k^{\gcd(m,n)} - 1$.

Alternatively, we could note if $n = qm + r$, then $\gcd(k^n - 1, k^m - 1) = \gcd(k^{n-m} - 1, k^m - 1) = \cdots = \gcd(k^{n-qm} - 1, k^m - 1) = \gcd(k^r - 1, k^m - 1)$. This strategy will be useful in the next question.

/9 pts **Problem 3.5.** Prove the Fibonacci numbers have strong divisibility.
(*Hint:* Show that if $n = qm + r$, then $\gcd(F_n, F_m) = \gcd(F_m, F_r)$.)

Solution 18. Let $n = mq + r$ using the division algorithm. Using Problem 2.3,

$$F_n = F_{mq+r} = F_{mq+1}F_r + F_{mq}F_{r-1}.$$

Since $F_m \mid F_{mq}$, we can subtract multiples of F_n using the Euclidean algorithm:

$$\gcd(F_n, F_m) = \gcd(F_{mq+1}F_r + F_{mq}F_{r-1}, F_m) = \gcd(F_{mq+1}F_r, F_m).$$

Finally, $\gcd(F_{mq+1}, F_m) = 1$ since consecutive Fibonacci numbers are coprime[1]:

$$\gcd(F_n, F_m) = \gcd(F_m, F_r).$$

This is the Euclidean algorithm! For example, if $m = 182$ and $n = 65$,

$$\gcd(F_{182}, F_{65}) = \gcd(F_{65}, F_{52}) = \gcd(F_{52}, F_{13}) = F_{13}.$$

It follows that $\gcd(F_m, F_n) = F_{\gcd(m,n)}$. □

References

[1] A. J. Cole and A. J. T. Davie. *A Game Based on the Euclidean Algorithm and a Winning Strategy for It.* The Mathematical Gazette Vol. 53, No. 386.

[2] Arthur T. Benjamin and Jennifer J. Quinn. *Proofs that Really Count: The Art of Combinatorial Proof.* Mathematical Association of America, Dolciani Mathematical Expositions.

[1]The proof is induction, $\gcd(F_n, F_{n-1}) = \gcd(F_{n-1}, F_{n-2}) = \cdots = \gcd(F_2, F_1) = 1$.