You have **60** minutes to work in teams on these problems. You must show your work to get full points. Each piece of paper should be your work on one problem only. At the top of each page you submit, you should write your team name and ID number, and "Problem X Page Y of Z", where X is the problem number, Y is the current page, and Z is the total number of pages submitted for that problem.

1. **[10 points]** Call a binary string *encompassing* if by erasing some digits from the string, one can obtain any binary string of length 2026.
   (a) **[3 points]** Find, with proof, the minimum possible length N of an encompassing string.
   (b) **[7 points]** Find, with proof, the number of length N encompassing strings (where N is the answer to part (a) of the question).

2. **[10 points]** In acute scalene triangle ABC, the foot of the altitude from A is D. ℓ is a line passing through A, and the feet of the altitudes from B and C to ℓ are B' and C', respectively. Show that the line through B' perpendicular to AC and the line through C' perpendicular to AB intersect on AD.

3. **[10 points]** Justin and Eduardo play a game. There is a calculator with a display and two buttons: the first button replaces the number x on the display with $\log_2 x$ (unless $x \leq 0$ in which case it displays ERROR), and the second button replaces x with N^x, where N is a fixed positive integer greater than 2. Justin chooses the value of N, and a positive integer to be initially displayed on the calculator screen. What is the smallest integer k such that Eduardo can press a sequence of k buttons that result in the displayed number not being a positive integer, regardless of Justin's choices?

4. **[10 points]** Let S be a set of 2026 points in 3D space placed by Ana, such that no three points in S lie on a line. Ana chooses a starting point P_0 in S. Then Ana and Bob play a game on these points, with Bob moving first.
   (a) Bob chooses a point P_1 in S distinct from P_0 and draws the segment $L_1 = \overline{P_0P_1}$.
   (b) Ana then chooses a point P_2 in S distinct from P_1 and draws the segment $L_2 = \overline{P_1P_2}$, such that the lengths satisfy $|L_2| > |L_1|$.
   (c) The players continue to alternate: if the previous segment drawn was $L_k = \overline{P_{k-1}P_k}$, the next player chooses a point P_{k+1} in S distinct from P_k such that $|L_{k+1}| = |\overline{P_kP_{k+1}}| > |L_k|$.
   (d) A player who cannot make a valid move loses.

   Over all sets S of 2026 points that Ana can choose, what is the maximum number of starting points $P_0 \in S$ that she can pick so that she will have a winning strategy?

5. **[15 points]** Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(n+1)^2 - f(n)^2 = 2f(f(n)) + 1$ for all $n \in \mathbb{N}$. Here, $\mathbb{N}$ denotes the set of positive integers (greater than or equal to 1).

6. **[15 points]** Given acute scalene triangle ABC with circumcircle Ω, points A', B', and C' are chosen on lines BC, AC, and AB respectively such that AA', BB', and CC' are tangent to Ω. Ω_A is the circle passing through A' which is tangent to Ω at B, Ω_B is the circle passing through B' that is tangent to Ω at C, and Ω_C is the circle passing through C' that is tangent to Ω at A. Prove that Ω_A, Ω_B, and Ω_C share a point.

7. **[20 points]** Let p be a prime, and let $\mathbb{Z}/p\mathbb{Z} = \{0, 1, \ldots, p-1\}$ be the set of residues modulo p. For any two 2026-tuples $a = (a_1, \cdots, a_{2026}), b = (b_1, \cdots, b_{2026}) \in (\mathbb{Z}/p\mathbb{Z})^{2026}$, and an integer $1 \leq i \leq 2026$, set $g_i(a, b) \equiv a_i + b_i + \sum_{j=1}^{i-1} a_j b_{i-j} \mod p$ and $g(a, b) = (g_1(a, b), \cdots, g_{2026}(a, b)) \in (\mathbb{Z}/p\mathbb{Z})^{2026}$. Show that for any $p \neq 2027$, there exists a function

$$f : (\mathbb{Z}/p\mathbb{Z})^{2026} \to \mathbb{Z}/p\mathbb{Z}$$

such that for any $a, b \in (\mathbb{Z}/p\mathbb{Z})^{2026}$, we have that

$$f(g(a, b)) \equiv f(a) + f(b) + \sum_{i=1}^{2026} a_i b_{2027-i} \mod p.$$