Chapter 3

Functions between Sets

3.1 Functions

3.1.1 Functions, Domains, and Co-domains

In the previous chapter, we investigated the basics of sets and operations on sets. In this chapter, we will analyze the notion of *function* between two sets. Similar to the functions from Pre-Calculus or Calculus, a function f will, to every input x, assign an output $f(x)$. In previous Mathematics courses, though, the functions we dealt with had as their inputs and outputs real numbers. In general, though, these inputs and outputs need to only be elements of two (perhaps different) sets.

Given two sets S and T, a **function** f from S to T (written $f : S \to T$) is an assignment, to every $s \in S$, one element $f(s) \in T$. Implicit in this definition are the following important properties of functions:

· Functions are well-defined. In other words, given an input $s \in S$, $f(s)$ takes on only a single value in T. In other words, if $f(s) = t_1$ and $f(s) = t_2$, then $t_1 = t_2$.

· Every input $s \in S$ has some output. Thus, we cannot have an $s \in S$ such that $f(s)$ has no value.

· For every $s \in S$, $f(s)$ must be an element of T. In other words, a function can only output elements of the output set T.

If $f : S \to T$ is a function, then the input set S is called the **domain** of f and the output set T is called the **co-domain** or **range**.

3.1.2 Examples of Functions

Below are some examples of functions between sets:

· Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Functions like this are the kind that were investigated in high school Mathematics courses. Here, the domain and co-domain are both the set of real numbers $\mathbb{R}$.

· Consider $f : \mathbb{R} \to \mathbb{Z}$ given by the *floor function* $f(x) = \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Clearly, $f(x) = \lfloor x \rfloor$ is defined for any $x \in \mathbb{R}$, and it will output an integer. Notice, that we could have also considered this f to be a function $f : \mathbb{R} \to \mathbb{R}$, since $\mathbb{Z} \subset \mathbb{R}$.

- Let $\mathbb{N}_+$ be the set of all positive integers: $\mathbb{N}_+ = \{1, 2, 3, 4, \ldots\}$ and consider the function $\varphi : \mathbb{N}_+ \to \mathbb{N}_+$, where $\varphi(n)$ is equal to the number of all positive integers less than or equal to n that share no common divisors with n. So, for example, $\varphi(9) = 6$ since the six numbers $1, 2, 4, 5, 7, 8$ share no common divisors with 9. Clearly, φ only makes sense for positive integers (hence the domain is $\mathbb{N}_+$) and can only output positive integers. Notice that, if p is prime, then $\varphi(p) = p - 1$ since all numbers $1 \leq n < p$ have no common factors with the prime p. This function φ is known as *Euler's totient function* and is of crucial importance in Number Theory.

- Consider the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $f(x, y) = x \cdot y$. Notice that this function inputs an element of the product $\mathbb{Z} \times \mathbb{Z}$ given by (x, y). Since $x, y \in \mathbb{Z}$, then the output $xy \in \mathbb{Z}$ and so $\mathbb{Z}$ is an appropriate co-domain for f.

- Let A be the set of all letters in the English alphabet. Thus

  $$A = \{a, b, c, \ldots, x, y, z\}.$$

  Consider the function $f : A \to \mathbb{N}$ that assigns to the n-th letter of the alphabet, the number n. Thus, $f(a) = 1$ and $f(z) = 26$. This function is well-defined for any letter in the alphabet and will output a natural number (since its place in alphabet is a non-negative whole number).

3.1.3 Images and Pre-images

If $f : S \to T$ and $f(s) = t$, then we say that the element t is the **image** of the element s. If we collect the images of every $s \in S$ into a set, that subset of T is called the **image** of f and is given by

$$\mathrm{Im}(f) = \{t \in T \mid \exists s \in S \text{ such that } f(s) = t\}.$$

Clearly, the image of the function f is a subset of the co-domain T. Thus, $\mathrm{Im}(f) \subset T$. However, it is not always the case that the image is the entire set T.

If $f(s) = t$, then we can describe this relation in another way. Since s maps to t, we can say that s is a **pre-image** of the element t. If we collect all the pre-images of a given t into a set, we can form the pre-image of t and define it as

$$f^{-1}(\{t\}) = \{s \in S \mid f(s) = t\}.$$

We may generalize this further and define the pre-image of a subset U of T. For this, the pre-image will be all elements in S that map into the subset U. Thus,

$$f^{-1}(U) = \{s \in S \mid f(s) \in U\}.$$

Below are some examples of images and pre-images of the functions given above.

- For $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$, the image of f is equal to

  $$\mathrm{Im}(f) = \{x \in \mathbb{R} \mid x \geq 0\},$$

  since every single non-negative number is outputted. As an example of a pre-image for this function, we have

  $$f^{-1}(\{3\}) = \{-\sqrt{3}, \sqrt{3}\}$$

$$f^{-1}(\{-5\}) = \varnothing.$$

Thus, the pre-image of an element may be empty. In fact, the pre-image of an element is non-empty if and only if that element is in the image of the function. We can also compute the pre-image of a set. For example,

$$f^{-1}((-2,6)) = (-\sqrt{6}, \sqrt{6}).$$

- For the floor function $f : \mathbb{R} \to \mathbb{Z}$ given by $f(x) = \lfloor x \rfloor$, notice that $\mathrm{Im}(f) = \mathbb{Z}$ since every single integer is the image of at least one (in fact many) real numbers. If we look at the pre-image of a single integer, say 4, we get that $f^{-1}(\{4\}) = [4, 5)$, since all numbers from 4 to 5 (including 4 but not 5) have that the largest integer less than or equal to it is 4.

- For the function $f : A \to \mathbb{N}$ that assigns to the n-th letter of the alphabet the number n, we see that its image is the integers between 1 and 26. Thus, $\mathrm{Im}(f) = \{n \in \mathbb{N} \,|\, 1 \leq n \leq 26\}$. If we investigate pre-images of individual elements in $\mathbb{N}$, we see that every n has at most 1 pre-image. In fact, $f^{-1}(\{n\})$ has 1 element if and only if $1 \leq n \leq 26$, and has 0 pre-images if $n = 0$ or $n > 26$. Functions like this, where individual elements have at most 1 pre-image, are known as *injective* functions and have the property that distinct element in the domain are sent to distinct elements in the co-domain.

3.1.4 Proofs about Images and Pre-images

Images and, in particular, pre-images are of importance in various fields of Mathematics because they behave well under the various set theory operations. In fact, pre-images of sets are used in Topology to define the continuity of functions between general sets.

Recall that if $f : S \to T$ is a function and $U \subset T$, then the pre-image of U under f is given by

$$f^{-1}(U) = \{s \in S \,|\, f(s) \in U\}.$$

By saying that pre-images behave well under the various set theory operations, we mean the following. If $U, V \subset T$, then

$$f^{-1}\left(\overline{U}\right) = \overline{f^{-1}(U)}$$

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$
$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V).$$

Key to the above proves is the equivalence of $x \in f^{-1}(U)$ and $f(x) \in U$. Below, we will prove the first two set equalities.

Proposition. Let $f : S \to T$ be a function and $U \subset T$. Then,

$$f^{-1}(\overline{U}) = \overline{f^{-1}(U)}.$$

Discussion.

`What we want:` $f^{-1}\left(\overline{U}\right) = \overline{f^{-1}(U)}$. Thus, we want to show two subset inclusions: $f^{-1}\left(\overline{U}\right) \subset \overline{f^{-1}(U)}$ and $\overline{f^{-1}(U)} \subset f^{-1}\left(\overline{U}\right)$.

`What we'll do:` For the first subset inclusion $f^{-1}\left(\overline{U}\right) \subset \overline{f^{-1}(U)}$, we will take an element $x \in f^{-1}\left(\overline{U}\right)$ and show that $x \in \overline{f^{-1}(U)}$. Because $x \in f^{-1}\left(\overline{U}\right)$, we know that $f(x) \in \overline{U}$. We will use this to show that $x \in \overline{f^{-1}(U)}$ by showing that $x \notin f^{-1}(U)$.

For the opposite inclusion $\overline{f^{-1}(U)} \subset f^{-1}\left(\overline{U}\right)$, we will take an $x \in \overline{f^{-1}(U)}$. Thus, $x \notin f^{-1}(U)$, and we will show that $x \in f^{-1}\left(\overline{U}\right)$ by showing that $f(x) \in \overline{U}$.

Proof. To show that $f^{-1}\left(\overline{U}\right) = \overline{f^{-1}(U)}$, we will show the following two subset inclusions: $f^{-1}\left(\overline{U}\right) \subset \overline{f^{-1}(U)}$ and $\overline{f^{-1}(U)} \subset f^{-1}\left(\overline{U}\right)$.

For the first subset inclusion, let $x \in f^{-1}\left(\overline{U}\right)$. Thus, $f(x) \in \overline{U}$ and $f(x) \notin U$. Since $f(x) \notin U$, then x is not mapped into U but is mapped into T; so $x \notin f^{-1}(U)$. Thus, $x \in \overline{f^{-1}(U)}$. Thus, $f^{-1}\left(\overline{U}\right) \subset \overline{f^{-1}(U)}$.

For the second subset inclusion, let $x \in \overline{f^{-1}(U)}$. Thus, $x \notin f^{-1}(U)$. So, it is false that $f(x) \in U$. So, $f(x) \notin U$ and thus $f(x) \in \overline{U}$. Thus, by definition $x \in f^{-1}\left(\overline{U}\right)$.

Thus, since we have shown both subset inclusions, we can conclude that $f^{-1}(\overline{U}) = \overline{f^{-1}(U)}$.

□

Proposition. Let $f : S \to T$ be a function and $U, V \subset T$. Then,

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V).$$

Discussion.

`What we want:` $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$. Thus, we wish to show the following two subset inclusions: $f^{-1}(U \cup V) \subset f^{-1}(U) \cup f^{-1}(V)$ and $f^{-1}(U) \cup f^{-1}(V) \subset f^{-1}(U \cup V)$.

`What we'll do:` For the first subset inclusion $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$, we will assume that $x \in f^{-1}(U \cup V)$, thus $f(x) \in U \cup V$. Since $f(x)$ is in a union, we can break it up into two cases: $f(x) \in U$ or $f(x) \in V$. In each case, we will conclude that $x \in f^{-1}(U) \cup f^{-1}(V)$.

For the second inclusion, $f^{-1}(U) \cup f^{-1}(V) \subset f^{-1}(U \cup V)$, we will assume that $x \in f^{-1}(U) \cup f^{-1}(V)$ and thus have the following two cases: $x \in f^{-1}(U)$ or $x \in f^{-1}(V)$. In each of the two cases, we will conclude that $x \in f^{-1}(U \cup V)$ by showing that $f(x) \in U \cup V$.

Proof. To prove that $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$, we will prove the two subset inclusions $f^{-1}(U \cup V) \subset f^{-1}(U) \cup f^{-1}(V)$ and $f^{-1}(U) \cup f^{-1}(V) \subset f^{-1}(U \cup V)$.

For the first subset inclusion, assume $x \in f^{-1}(U \cup V)$; thus, we have that $f(x) \in U \cup V$. This means that $f(x) \in U$ or $f(x) \in V$ and we have two cases. If $f(x) \in U$, then $x \in f^{-1}(U)$ and thus $x \in f^{-1}(U) \subset f^{-1}(U) \cup f^{-1}(V)$. For the second case, we have that $f(x) \in V$ and thus $x \in f^{-1}(V)$. Thus, $x \in f^{-1}(V) \subset f^{-1}(U) \cup f^{-1}(V)$. In either case, we have that $x \in f^{-1}(U) \cup f^{-1}(V)$.

For the second subset inclusion, let $x \in f^{-1}(U) \cup f^{-1}(V)$. Thus, $x \in f^{-1}(U)$ or $x \in f^{-1}(V)$, giving us two cases. In the first case, $x \in f^{-1}(U)$ and thus $f(x) \in U$. So, $f(x) \in U \subset U \cup V$ and thus $f(x) \in U \cup V$. This gives us that $x \in f^{-1}(U \cup V)$. In the second case, $x \in f^{-1}(V)$ and thus $f(x) \in V$. So, $f(x) \in V \subset U \cup V$ and thus $f(x) \in U \cup V$. This gives that $x \in f^{-1}(U \cup V)$. In either case, $x \in f^{-1}(U \cup V)$ and so we have the subset inclusion $f^{-1}(U) \cup f^{-1}(V) \subset f^{-1}(U \cup V)$.

By proving both subset inclusions, we can conclude that $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$.

□

3.2 Injections, Surjections, and Bijections

In general, functions between sets can get very complicated. In this section, we investigate two types of functions which have particularly nice properties with respect to their images and pre-images.

3.2.1 Definitions of an Injection and a Surjection

One part of the definition of a function is that for every element in the domain $s \in S$, there exists some $f(s) \in T$. That means that every s gets maps to some t and thus $f(s) = t$. However, in many functions, there are multiple different values for s that will output exactly the same t. For example, in the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$, $f(-3) = 9$ and $f(3) = 9$. Functions where this does not happen are particularly nice and are called *injective* or *one-to-one* functions. These kinds of functions have the property that if s_1 and s_2 are distinct elements, then their outputs $f(s_1)$ and $f(s_2)$ are also distinct elements. In other words, if $s_1 \neq s_2$, then $f(s_1) \neq f(s_2)$. Given that there are many negative statements in this definition, we can take the equivalent contrapositive and arrive at a more helpful formulation. So, a function $f : S \to T$ is called **injective** or **one-to-one** if whenever $f(s_1) = f(s_2)$, then $s_1 = s_2$.

Another concern is that even though every $s \in S$ must have some output $f(s) \in T$, it is not guaranteed that every $t \in T$ will have some pre-image. In other words, we always know that $\text{Im}(f) \subset T$, but it is not always true that $\text{Im}(f) = T$. In the case where the image is the entire co-domain T (in other words, when every t has some pre-image), our function is called *surjective* or *onto*. More formally, a function $f : S \to T$ is called **surjective** or **onto** if for every $t \in T$, there exists some $s \in S$ such that $f(s) = t$.

Below are some examples of functions and a discussion about their injectivity and surjectivity.

- The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not injective since, as stated above, $f(-3) = 9 = f(3)$. Thus, it is not true that whenever $f(s_1) = f(s_2)$, then $s_1 = s_2$. In fact, if we tried to prove injectivity, we would assume that $f(s_1) = f(s_2)$ and thus $s_1^2 = s_2^2$; however, we would not be able to conclude that $s_1 = s_2$. As for surjectivity, $f(x) = x^2$ fails this as well. Notice that -5 has no pre-image since there is no x such that $f(x) = x^2 = -5$. Non-surjectivity is also clear because $\text{Im}(f) = \{x \in \mathbb{R} \mid x \geq 0\}$, which is not the entire co-domain $\mathbb{R}$.

- The floor function $f : \mathbb{R} \to \mathbb{Z}$ given by $f(x) = \lfloor x \rfloor$ is not injective. To see this, note that $f(3) = 3 = f(\pi)$. In fact, non-injectivity should be clear by the fact that the pre-image of a single natural number a will be the set $f^{-1}(\{a\}) = [a, a+1)$, which contains infinitely many points. If f were injective, then the pre-image of a point would contain at most 1 point. The floor function is indeed surjective. To show this, if we take an arbitrary element in the co-domain $a \in \mathbb{Z}$, then the real number a maps to a. In other words, $f(a) = \lfloor a \rfloor$ and thus every a has at least one pre-image. Surjectivity is also apparent because the pre-image of every element in the co-domain is non-empty. Notice that, if we had presented the floor function $f : \mathbb{R} \to \mathbb{R}$ as a function with a co-domain of $\mathbb{R}$, then the function would no longer be surjective because not every real number has a pre-image.

- Recall the function $f : A \to \mathbb{N}$, where A is the English alphabet, where f takes the n-th letter of the alphabet to $n \in \mathbb{N}$. This function f is injective because for every $n \in \mathbb{N}$, there are either no pre-images (for example, there is no 27-th letter of the alphabet) or there is 1 pre-image (for example, there is only one 4-th letter, d). The function f is not surjective, though, because, for example, 27 has no pre-image (as there is no 27-th letter of the alphabet).

3.2.2 Proofs involving Injections and Surjections

The definition of an injection and a surjection gives us a framework for how to go about proving that a function is or is not injective or surjective. We give examples below.

- Consider the function $f : \mathbb{R} \to \mathbb{R}_+$ given by $f(x) = e^x$, where $\mathbb{R}_+$ is the set of all positive real numbers:

  $$\mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x > 0\}.$$

  We will show that $f(x)$ is a surjection. To do this, we must take an arbitrary element in the co-domain $t \in \mathbb{R}_+$ and find a pre-image in the domain $s \in \mathbb{R}$. Thus, s must satisfy the equation $f(s) = e^s = t$. If we let $s = \ln t$, then the equation should hold. Note that, since $t > 0$, we are indeed allowed to take its logarithm. Thus, every $t \in \mathbb{R}_+$ has the pre-image $\ln t \in \mathbb{R}$.

- We can also show that the $f : \mathbb{R} \to \mathbb{R}_+$ given above by $f(x) = e^x$ is also injective. To do so, we will assume that $f(s_1) = f(s_2)$ and conclude that $s_1 = s_2$. Thus, we have that $e^{s_1} = e^{s_2}$. Taking logarithms of both sides (which we are allowed to do since $e^s > 0$ for any s), we have that $\ln e^{s_1} = \ln e^{s_2}$ and thus $s_1 = s_2$, as desired.

- Consider again the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $f(m, n) = m \cdot n$. The above function is clearly surjective since every $n \in \mathbb{Z}$ has a pre-image of $(n, 1) \in \mathbb{Z} \times \mathbb{Z}$ (notice that $(1, n)$ is also a valid pre-image). Injectivity, on the other hand, fails. To show that a function is not injective, we need only provide two distinct numbers that map to the same element. Take, for example, $(6, 4)$ and $(2, 12)$; notice that $f(6, 4) = 24 = f(2, 12)$, and thus f is not injective. In fact, although this is not necessary, we can show that *any* integer has multiple pre-images.

- Consider the function $f : \mathbb{R}_{\neq 0} \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$, where $\mathbb{R}_{\neq 0}$ is the set of all non-zero real numbers. First, note that we chose our domain of $\mathbb{R}_{\neq 0}$ because the function is not defined at 0, but is defined at all other reals. This function is injective, since if $f(x) = f(y)$, then $\frac{1}{x} = \frac{1}{y}$, and, cross-multiplying, we get that $x = y$. The function f, though, is not surjective since $0 \in \mathbb{R}$ has no pre-image because $f(x) = \frac{1}{x} = 0$ has no solution. It is important to point out, though, that if we had instead thought of f as a function $f : \mathbb{R}_{\neq 0} \to \mathbb{R}_{\neq 0}$, then f would remain injective and would also become surjective.

Injections and surjections behave particularly nicely under *composition of functions*. Given two functions $f : S \to T$ and $g : T \to R$, we can form the **composition function** $g \circ f : S \to R$ given by

$$g \circ f(s) = g(f(s)).$$

This makes sense because, if $s \in S$, then $f(s) \in T$ and thus we can apply the function g to it, giving us $g(f(s))$, which is an element of R.

We will now show that if f and g are both injective (resp., surjective), then $g \circ f$ is also injective (resp., surjective). For injectivity, this is intuitively clear since f maps distinct elements $s_1 \neq s_2$ to distinct elements $f(s_1) \neq f(s_2)$; since g is also injective, the distinct element $f(s_1) \neq f(s_2)$ are sent to distinct element $g(f(s_1)) \neq g(f(s_2))$. Surjectivity follows the same scheme: any $r \in R$ has a pre-image in T, which is turn has a pre-image in S; this pre-image in S will be the pre-image of r in the composition.

Proposition. Let S, T, and R be sets with functions $f : S \to T$ and $g : T \to R$. If f and g are both injective functions, then the composition $g \circ f : S \to R$ is also injective.

Discussion.

`What we know:`

- $g : T \to R$ is injective. Thus, whenever we know that $g(t_1) = g(t_2)$, we can conclude that $t_1 = t_2$.
- $f : S \to T$ is injective. Thus, whenever we know that $f(s_1) = f(s_2)$, we can conclude that $s_1 = s_2$.

`What we want:` $g \circ f : S \to R$ is injective. Thus, we will assume that $g(f(s_1)) = g(f(s_2))$ and show that $s_1 = s_2$.

`What we'll do:` To show the above, we will first use the injectivity of g with $f(s_1)$ playing the role of t_1 and $f(s_2)$ playing the role of t_2. Then, we will apply the injectivity of f.

Proof. We will show that $g \circ f$ is injective by showing that whenever $g(f(s_1)) = g(f(s_2))$ for $s_1, s_2 \in S$, we can conclude that $s_1 = s_2$.

If $g(f(s_1)) = g(f(s_2))$, then, since g is injective, its inputs are equal: $f(s_1) = f(s_2))$. Since f is injective and $f(s_1) = f(s_2)$, then $s_1 = s_2$, as desired. Thus, the composition function $g \circ f$ is injective.

□

Proposition. Let S, T, and R be sets with functions $f : S \to T$ and $g : T \to R$. If f and g are both surjective, then the composition function $g \circ f : S \to R$ is also surjective.

Discussion.

`What we know:`

- $g : T \to R$ is surjective. Thus, for any $r \in R$, there exists a $t \in T$ such that $g(t) = r$.
- $f : S \to T$ is surjective. Thus, for any $t \in T$, there exists an $s \in S$ such that $f(s) = t$.

`What we want:` $g \circ f : S \to R$ is surjective. So, we wish to show that for any $r \in R$, there exists an $s \in S$ such that $g(f(s)) = r$.

`What we'll do:` We will let $r \in R$ and use the surjectivity of g to find a pre-image $t \in T$ so that $g(t) = r$. Then, we will use the surjectivity of f to find a pre-image $s \in S$ for t so that $f(s) = t$. We will then check that, indeed $g(f(s)) = r$.

Proof. To show that $g \circ f$ is surjective, we will take an element $r \in R$ and find an $s \in S$ such that $g(f(s)) = r$.

Let $r \in R$. Since g is surjective, there exists a $t \in T$ such that $g(t) = r$. Since $t \in T$ and f is surjective, there exists an $s \in S$ such that $f(s) = t$. Notice that this s is a pre-image for r under the composition function since $g(f(s)) = g(t) = r$. Thus, $g \circ f$ is surjective.

□

3.3 Bijections

On its own, an injection or a surjection is a very special kind of function. When a function enjoys both properties, it exhibits a very rigid structure and gives us much information about the domain and co-domain S and T. So, we define a function $f : S \to T$ to be a **bijection** if it is both an injection and a surjection.

If f is a surjection, then for every $t \in T$, there exists *at least one pre-image* $s \in S$ such that $f(s) = t$. On the other hand, if f is a injection, then every $t \in T$ has *at most one pre-image* $s \in S$ such that $f(s) = t$. In a bijection, both of these properties hold, so we have that for every $t \in T$, there is *exactly one pre-image*. So, a bijection $f : S \to T$ gives a correspondence between S and T that makes them, in terms of their set-theoretic properties, identical.

In terms of the sizes or *cardinalities* of the sets S and T, if $|S|$ gives the number of element in S and $|T|$ gives the number of elements in T, then we may say the following about the relative sizes of S and T if S and T are finite sets.

- If $f : S \to T$ is an injection, then there must be at least as many elements in T as there are in S since distinct elements in S must map to distinct elements in T. So, $|S| \leq |T|$.

- If $f : S \to T$ is a surjection, then there must be at least as many element in S as there are in T since every $t \in T$ needs at least one pre-image. Thus, $|S| \geq |T|$.

- If $f : S \to T$ is a bijection, then, combining the two inequalities above, the number of elements in S must be equal to the number of elements in T: $|S| = |T|$.

The above observations are clear for finite sets and they, in fact, give a formal way for mathematicians to discuss cardinalities of sets even when the two sets are infinite.

Below are some examples of bijections.

- Let $S = \{a, b, c\}$ and $T = \{1, 2, 3\}$. The function $f : S \to T$ given by $f(a) = 2, f(b) = 1$, and $f(c) = 3$ can be checked to be a bijection. Notice that $|S| = 3 = |T|$, where the equality of cardinalities is a necessary condition for the existence of a bijection.

- The function $f : \mathbb{R} \to \mathbb{R}_+$ given by $f(x) = e^x$ is a bijection. Both injectivity and surjectivity were shown above.

- The function $f : \mathbb{R}_+ \to \mathbb{R}$ given by $f(x) = \ln x$ is a bijection as well. For surjectivity, note that if $a \in \mathbb{R}$, then $e^a \in \mathbb{R}_+$ is its pre-image since $f(e^a) = \ln e^a = a$. For injectivity, note that if $f(s_1) = f(s_2)$, then $\ln(s_1) = \ln(s_2)$ and placing both sides of the equation in exponents base e yields $e^{\ln s_1} = e^{\ln s_2}$, leaving us with our desired equation $s_1 = s_2$. Thus, f is both surjective and injective and thus a bijection.

- The function $f : \mathbb{R}_{\neq 0} \to \mathbb{R}_{\neq 0}$ given by $f(x) = \frac{1}{x}$ is a bijection. We previously proved that f was injective. We also proved that f was not surjective, but this is when we had a co-domain of $\mathbb{R}$. With the co-domain of $\mathbb{R}_{\neq 0}$, we can see that f is indeed surjective since $a \in \mathbb{R}_{\neq 0}$ has a pre-image of $\frac{1}{a}$ because

$$f\left(\frac{1}{a}\right) = \frac{1}{1/a} = a.$$

Note that if we had kept $\mathbb{R}$ as our co-domain, our above proof would have failed because $\frac{1}{a}$ is not defined if $a \neq 0$.

- For any set S, the *identity function* $i_S : S \to S$ given by $i_S(s) = s$ for all $s \in S$ is a bijection. This function is called the identity because the output s is identical to the input s. Injectivity comes from noting that if $i_S(s_1) = i_S(s_2)$, then, by definition of i_S, $s_1 = s_2$. Surjectivity is also clear since if $s \in S$, then it is its own pre-image since $i_S(s) = s$. If $S = \mathbb{R}$, then the identity function on $\mathbb{R}$ is just the linear function $f(x) = x$.

Since composition of functions behave so nicely with respect to injectivity and surjectivity, it stands to reason that the composition of two bijections is a bijection. Indeed, this is the case, and the proof is straightforward given that we have already shown that that injectivity and surjectivity persist under composition of functions.

Theorem. Let $f : S \to T$ and $g : T \to R$ be bijections. Then, $g \circ f : S \to R$ is also a bijection.

Proof. To show that $g \circ f$ is a bijection, we must show that it is both injective and surjective. Since f and g are bijections, they are both injective. Thus, by our previous proposition, $g \circ f$ is also injective. Furthermore, since f and g are bijections, they are both surjective. Thus, by our previous proposition, $g \circ f$ is also surjective. Thus, $g \circ f$ is both injective and surjective and thus a bijection. $\square$

3.3.1 Invertibility of Bijections

In the previous examples, we saw that the two functions e^x and $\ln x$ were both bijections, the former as a function $\mathbb{R} \to \mathbb{R}_+$ and the latter as a function $\mathbb{R}_+ \to \mathbb{R}$. In fact, given that the function e^x and $\ln x$ are known to "undo" each other, a natural question to ask is, if a given $f : S \to T$ is a bijection, is its *inverse function* $f^{-1} : T \to S$ a bijection as well?

Before we begin to answer the above question, we first need to see under what conditions a function $f : S \to T$ is invertible. For such a function, its **inverse function** is defined to be a function $f^{-1} : T \to S$ such that

$$f^{-1}(f(s)) = s$$

for all $s \in S$. If we attempt to define such a function f^{-1}, a good first attempt would be as follows: if $f(s) = t$, then define $f^{-1}(t) = s$. Such a function would certainly have the desired property of being an inverse since $f^{-1}(f(s)) = f^{-1}(t) = s$. Since we are defining this function from scratch, we must be sure it is well-defined. The first concern is that, if f^{-1} is to be defined for every $t \in T$, then we must need for the equation $f(s) = t$ to work for at least one s. In other words, f must be a surjection. Similarly, we may have the problem that two distinct element $s_1 \neq s_2$ could map to t: $f(s_1) = t = f(s_2)$; if this occurred, then it would not be clear if we should define $f^{-1}(t)$ to be s_1 or s_2. If, however, f is injective, then f^{-1} would be well-defined since whenever $f^{-1}(t) = s_1$ and $f^{-1}(t) = s_2$, then by definition, $f(s_1) = t = f(s_2)$ and, by the injectivity of f, $s_1 = s_2$.

The above argument shows that if $f : S \to T$ is a bijection, then its inverse $f^{-1} : T \to S$ can be defined. This still leaves us with the question of if f^{-1} is also a bijection. The below proof shows that this is indeed true.

Proposition. Let $f : S \to T$ be a bijection. Then, its inverse function $f^{-1} : T \to S$ given by $f^{-1}(t) = s$ if $f(s) = t$ is also a bijection.

Discussion. Key to the below discussion is the interplay between $f : S \to T$ and $f^{-1} : T \to S$. Since f is a bijection, we already know that its inverse f^{-1} will be well-defined, and the relationship is given by $f^{-1}(t) = s$ if and only if $f(s) = t$.

`What we know:` We know that f is a bijection. This is needed because this tells us that f^{-1} exists. More subtle, but very important, is that $f : S \to T$ is a well-defined function. Thus, f enjoys the following properties.

- Every $s \in S$ has some output $f(s) = t \in T$.
- If $f(s) = t_1$ and $f(s) = t_2$, then $t_1 = t_2$.

`What we want:` We wish to show that $f^{-1} : T \to S$ is a bijection. Thus, we will show that

- f^{-1} is injective. Thus, we need to show that whenever $f^{-1}(t_1) = f^{-1}(t_2)$, then $t_1 = t_2$.
- f^{-1} is surjective. Thus, for every $s \in S$, we must show that there exists some $t \in T$ such that $f^{-1}(t) = s$.

`What we'll do:` First, we will state that the function $f^{-1} : T \to S$ given in the proposition is well-defined by our above arguments. To show injectivity of f^{-1}, we will assume that $f^{-1}(t_1) = f^{-1}(t_2)$ and use the fact that f is well-defined to show that $t_1 = t_2$. To show surjectivity, we will find a pre-image under f^{-1} for $s \in S$ by noting that, as f is a function, $f(s)$ is equal to some $t \in T$.

Proof. Since f is a bijection, our previous discussion indicates that the inverse function $f^{-1} : T \to S$ given by $f^{-1}(t) = s$ when $f(s) = t$ is well-defined. We now need to show that f^{-1} is both injective and surjective for it to be a bijection.

For injectivity, assume that $f^{-1}(t_1) = f^{-1}(t_2)$; we will show that $t_1 = t_2$. Give the common element $f^{-1}(t_1) = f^{-1}(t_2)$ the name $s \in S$. Thus, $f^{-1}(t_1) = s$ and $f^{-1}(t_2) = s$. Then, by definition of f^{-1}, $f(s) = t_1$ and $f(s) = t_2$. Since f is a well-defined function, we can conclude that $t_1 = t_2$, as desired. Thus, f^{-1} is injective.

For surjectivity, let $s \in S$. Consider the element $f(s) = t \in T$. Since f is a function, such a t exists. This t is the pre-image of s under f^{-1} since the equation $f(s) = t$ implies that $f^{-1}(t) = s$. Thus, f^{-1} is surjective.

Since f^{-1} was shown to be a well-defined function that is both injective and surjective, it is a bijection.

$\square$

Thus, if we ever know that $f : S \to T$ is a bijection, then we should be able to figure out its inverse $f^{-1} : T \to S$. We do so below for the previous bijection examples.

- The function $f : \{a, b, c\} \to \{1, 2, 3\}$ given by $f(a) = 2, f(b) = 1$, and $f(c) = 3$ is clearly a bijection, and its inverse $f^{-1} : \{1, 2, 3\} \to \{a, b, c\}$ is given by
$$f^{-1}(1) = b,\ f^{-1}(2) = a,\ f^{-1}(3) = c,$$
which is also a bijection.

- For $f : \mathbb{R} \to \mathbb{R}_+$ given by $f(x) = e^x$. its inverse is $f^{-1} : \mathbb{R}_+ \to \mathbb{R}$ given by $f(x) = \ln x$, a bijection.
- For $f : \mathbb{R}_+ \to \mathbb{R}$ given by $f(x) = \ln x$, its inverse is $f^{-1} : \mathbb{R} \to \mathbb{R}_+$ given by $f^{-1}(x) = e^x$. The relationship of this inverse to the previous example's inverse is generally true: if f is a bijective function and f^{-1} is its inverse, then the inverse of f^{-1} is the original function f.
- For $f : \mathbb{R}_{\neq 0} \to \mathbb{R}_{\neq 0}$, given by $f(x) = \frac{1}{x}$, its inverse is itself: $f^{-1}(x) = \frac{1}{x}$. One can see that this is indeed the inverse since

$$f^{-1}(f(x)) = f^{-1}\left(\frac{1}{x}\right) = \frac{1}{1/x} = x.$$

- For the identity function $i_S : S \to S$, the inverse of the identity is again the identity. Thus, $i_S^{-1}(s) = s$. To verify, we can see that

$$i_S^{-1}\left(i_S(s)\right) = i_S^{-1}(s) = s.$$

Note that the notation for an inverse is almost identical to the notation for the pre-image of a function. The distinction is important, but subtle. If $f : S \to T$ is a bijection (and thus invertible), and $f(s) = t$, then its inverse is given by $f^{-1}(t) = s$, and the pre-image of t is given by $f^{-1}(\{t\}) = \{s\}$. The difference is that the first equation is describing a function while the second equation is describing set equality.