1. Interpret the regular expression.
   (a) Strings with exactly one 1.
   (b) Strings of even length.
   (c) Strings in which 0s occur only in blocks of 3.

2. For these, especially part (c), equivalent answers may exist. Grade accordingly; compile a list of common correct answers as you go.
   (a) $\Sigma^*1\Sigma^*$
   (b) $0\Sigma^*0 \cup 1\Sigma^*1$
   (c) $\Sigma(0^*10^*10^*)^*$

3. Generating function basics
   (a) Rewriting our sequence as a generating function gives $1 + 0x + 3x^2 + 0x^3 + 9x^4 + \dots$ We see that this is a geometric series starting with 1 with common ratio $3x^2$, so our generating function in closed form is $\frac{1}{1-3x^2}$.
   (b) Each decomposition of an arbitrary integer n into 2 positive integers $k + (n-k) = n$ contributes $a_k b_{n-k}$ to the coefficient of x^n. Summing over all possible such decompositions gives us $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.
   (c) It is clear that $\frac{1}{1-x^c}$ is the generating function for c_n, the number of ways to make change with coins of value c cents. The generating function we're looking at is the product of the generating functions with $c \in \{1, 5, 10, 25\}$. By part (b), we see that the sequence for $\frac{1}{(1-x^{c_1})(1-x^{c_2})}$ is given by $(c_1c_2)_n = \sum_{k=0}^{n} a_k b_{n-k}$. This is indeed the number of ways to make change for n cents given coins of value c_1 and c_2.

4. Generating functions and recurrence relations
   (a) Let $f(x)$ be the generating function for the Fibonacci sequence. Then $f(x)$ satisfies $f(x) - xf(x) - x^2 f(x) = (F_1 - F_0)x + F_0$. Thus $f(x) = \frac{1}{1-x-x^2}$.
   (b) Letting $f(x) = \frac{1+x^2}{1+2x-x^3}$, $f(x) + 2xf(x) - x^3 f(x) = 1 + x^2$. Then coefficients of terms of degree $n \geq 3$ on the LHS must sum to zero, i.e. $F_n + 2F_{n-1} - F_{n-3} = 0$, which gives the desired recurrence.
   (c) We know that $(\Sigma a_n x^n)(c_0 + c_1 x + \cdots + c_k x^k) = p(x)$. For n sufficiently large ($n > d$), $0x^n = (a_n c_0 + a_{n-1} c_1 + \ldots + a_{n-k} c_k)x^n$. Therefore, $0 = a_n c_0 + a_{n-1} c_1 + \ldots + a_{n-k} c_k$.

5. Let the generating function representing A be $f_A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ and B be $f_B(x) = b_0 + b_1 x + b_2 x^2 + \ldots$. Then the number of strings of length n in $A \circ B$ is the number of strings of length n formed from stings of length 0 in A and length n in B, the strings of length 1 in A and those of length $n-1$ in B, etc. so we see that c_n must be the sum

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

   and thus our generating function for $A \circ B$ must be the product of the generating functions for A and B, $f_A(x) \cdot f_B(x)$.

6. As above, let the generating function representing A be $f_A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ and B be $f_B(x) = b_0 + b_1 x + b_2 x^2 + \ldots$. Consider the number of strings of length n in $A \cup B$. We can either have a string of length n from $f_A(x)$, of which we have a_n, or a string of length n from B, of which we have b_n. So we have $a_n + b_n$ ways to have a string of length n in our regular expression $A \cup B$, so our generating function must be $f_A(x) + f_B(x)$.

7. For any regular expression A, A^* consists of all of the product sets of elements in A, $A^0, A^1, A^2, \ldots$ We claim that the generating function of A^n is $[f_A(x)]^n$, where $f_A(x)$ is the generating function for A. We proceed by induction. Clearly the generating function of A is $f_A(x) = [f_A(x)]^1$, so this establishes our base case. Now assume for some $n \geq 1$ the statement holds: $[f_A(x)]^n$ is the generating function for A^n. Now consider the generating function for A_{n+1}: we can think of this as the concatenation of regular expressions A_n and A. By problem 6, this means our generating function must be $[f_A(x)]^{n+1}$. So our generating function in total is the union of all the generating functions for A^i; by problem 5 this is

$$1 + f_A(x) + [f_A(x)]^2 + [f_A(x)]^3 + \ldots$$

This a geometric series with first term 1 and common ratio $f_A(x)$, so our generating function for A^* is

$$\frac{1}{1 - f_A(x)}$$

8. Strings in which 0s appear only in odd blocks and 1s appear only in blocks of size 1 or 2.

9. Using the rules developed in (5-7), we can calculate the generating function to be

$$(1 + x + x^2)\frac{1}{1 - (\frac{1}{1-x^2})x(x + x^2)}\left(1 + \left(\frac{1}{1 - x^2}\right)x\right) = \frac{x^4 - x^2 - 2x - 1}{x^3 + 2x^2 - 1}$$

10. From 4(c) and the closed form found in 9, we know that a_n satisfies the recurrence relation:

$$-1a_n + 2a_{n-2} + a_{n-3} = 0$$