Primer to Mathematical Writing

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December 2022

Introduction

Ever since its outset, CHMMC has always included a *proof-based* aspect to the competition. Unfortunately, many high schoolers who attend the competition may either have very little experience with mathematical writing, or no experience at all. We intend to remedy this by providing **resources** and **tips** on the CHMMC website about the basics of mathematical writing. The **six chapters** of content on **Logic**, **Set Theory**, **Functions**, **Real Sets**, **Complex Numbers**, and **Induction**, posted on the **Resources** page of the CHMMC website, are more than enough for you to gain a basic familiarity with these concepts! The material is from the (unofficial) Caltech course: **Math 0: Transition to Mathematical Proofs**, targeted towards incoming Caltech students with no prior proof background. It aims to be a friendly, accessible introduction to proof-writing and prepare students for Caltech's intensive first-year coursework in mathematics.

The **Math 0** content is very detailed—the aim is to minimize ambiguity with any definition or concept introduced. We certainly do not expect students to show that much detail during the CHMMC. Moreover, not all concepts of these six chapters are created equal. Among the topics covered throughout these six chapters, the **most important** of them for the CHMMC are

1. **Chapter 1 (Logic)**: If, then statements. If and only if statements. Existence and uniqueness.
2. **Chapter 1 (Logic)**: Proof by contrapositive.
3. **Chapter 2 (Set Theory)**: Basic set theory. Subsets, Containment, Union, Intersection, Cartesian product. Proofs of set equality.
4. **Chapter 2 (Set Theory)**: Proof by contradiction.
5. **Chapter 3 (Functions)**: Definition of function, domain, codomain, range.
6. **Chapter 3 (Functions)**: Definition of injective, surjective, bijective.
7. **Chapter 3 (Functions)**: The general concept of "well-definedness."
8. **Chapter 4 (Real Sets)**: Integers, rational numbers, irrational numbers.
9. **Chapter 5 (Complex Numbers)**: Basic complex numbers. $e^{i\theta} = \cos\theta + i\sin\theta$. Complex polar form.
10. **Chapter 6 (Induction)**: Basic proof by induction. Also do some independent reading on Strong Induction.[1]

Remark: you do not need to worry about any parts of these Chapters that use Calculus!

[1] The principle of "Strong Induction" may in fact be deduced from the principle of "Weak Induction!"

We expect that students understand the concepts listed above "at the **basic level**" for the CHMMC Power Round. For instance, we will assume you can recite the definition of a bijection, and will be able to apply this definition to the Power Round (if appropriate). Beyond the scope of CHMMC, you might find this material very handy if you ever have to hone your proof-writing skills for higher math classes or the USA(J)MO/IMO. Or it might just be a new perspective for you to think about things, opening up broader ideas in the future. Indeed, the content of **Math 0** is only the very tip of the mathematical iceberg; we hope you eventually find the beauty in composing arguments and proofs across various areas of mathematics.

For more general tips on solution- and proof-writing, we direct you to the following two resources:

1. (AoPS, How to Write a Proof): `https://artofproblemsolving.com/blog/articles/how-to-write-a-solution`
2. (Evan Chen, Remarks on English): `https://web.evanchen.cc/handouts/english/english.pdf`

It would be a good idea to review these articles after reading the six chapters of posted content, as they are more closely tailored to the expectations on a high school competition (such as CHMMC). In particular, the solution write-ups in the AoPS article are a good representation of the **amount of detail** we expect students to give in their solutions and proofs on the CHMMC Power Round.

On another note, we urge students not to be discouraged if they find the either the **Math 0** lectures or the following exercises difficult, because, well, mathematical proof-writing is indeed **very challenging**! It will take A. Lot. of time and effort to improve your proof-writing skills. Here at CHMMC we are not looking for students to write perfect proofs, but we are expecting them to demonstrate a certain level of clarity in their solutions. There is no set "benchmark," but your writeups should be clear enough such that if you were to present it to one of your fellow team members, they would be able to understand your language relatively easily. We hope that you will never lose points on a problem you knew how to do because your solution was not understandable to the grader due to readability issues.

The above resources have already provided many many examples of proofs for you to *read*; so it is now up to you to *write* some proofs of your own. Indeed, the next few pages contain a collection of (mostly) "easy" exercises pertaining to the six chapters of posted content. When I say "easy," I mean that the arguments needed for these exercises should require familiar techniques and feel relatively straightforward with a **perfect understanding** of the required content. Of course, nothing is easy when first learning a subject; patience and perseverance are almost prerequisites to the study of mathematics itself! We hope that these exercises will bolster your understanding of these topics whether you have never done a mathematical proof before in your life or you are already well-versed in proof-writing.

Many of the exercises below are pulled directly from the HW from Caltech's **Math 0** program. You are free to do as many or few of these exercises as you wish; exercises marked with an $(*)$ are more difficult than the others and may be safely skipped. You may also get all the help you need to do these exercises (e.g. collaborating with your team members), but do beware that consulting a prepared solution to a problem before making a conscientious effort to do it yourself will not help you learn the material well. Moreover, to maximize your learning, we suggest you write up solutions to exercises pertaining to a specific chapter *without referencing the content or ideas of later chapters*. For practice, you may also include a "Discussion" section prior to each of your proofs, as do proofs of statements in the **Math 0** lectures.

1 Logic

1. Let $m \neq 0$ and b be real numbers. Show that there exists a unique x such that $mx+b=0$.
2. Let x be a real number. Prove that $-1 \leq x \leq 1$ if and only if $x^2 \leq 1$ (in proving this, it may be helpful to note that $-1 \leq x \leq 1$ is equivalent to $-1 \leq x$ and $x \leq 1$).
3. Two whole numbers are said to *have the same parity* if they are both even or both odd. Prove the following biconditional statement:

   Let m and n be whole numbers. m and n have the same parity if and only if $m+n$ is even.

2 Set Theory

1. In the notes, we proved one of DeMorgan's Set Theory laws. Prove the remaining one. That is, prove the following statement:

   Let S and T be sets. Then
   $$\overline{S \cap T} = \overline{S} \cup \overline{T}.$$

2. In the notes, we proved one distributive law. Prove the remaining one. That is, prove the following statement:

   Let S, T, and R be sets. Then,
   $$S \cap (T \cup R) = (S \cap T) \cup (S \cap R).$$

3. Let A, B, C, and D be sets. Show that if $A \subset B$ and $C \subset D$, then $A \times C \subset B \times D$.
4. $(*)$ Let A be a set, and $\mathscr{P}(A)$ be the collection of all subsets of A ($\mathscr{P}(A)$ is called the *power set* of A). For any subsets $X, Y \subset A$, define a binary operation $+$ on $\mathscr{P}(A)$ as follows:
   $$X+Y = X \cup Y \setminus (X \cap Y)$$
   (a) Prove the *associative property*: for $X, Y, Z \in \mathscr{P}(A)$,
   $$X + (Y+Z) = (X+Y) + Z$$
   (b) Prove the *distributive property*: For $C, X, Y \in \mathscr{P}(A)$,
   $$C \cap (X+Y) = (C \cap X) + (C \cap Y)$$

   It might be a good idea to approach this problem by "drawing Venn diagrams." Of course, a Venn diagram itself is not a proof.

3 Functions

1. Let $f : S \to T$ and $g : T \to S$ be functions.

   (a) Suppose $g \circ f : S \to S$ is the identity function, i.e. $g(f(x)) = x$ for all $x \in S$. Prove that f is injective. In this situation, g is called a *left inverse* of f.

(b) $(*)$ Suppose $f \circ g : T \to T$ is the identity function, i.e. $f(g(x)) = x$ for all $x \in T$. Do some research on the *Axiom of Choice* (on Wikipedia, for example). Prove that f is surjective, assuming the axiom of choice. In this situation, g is called a *right inverse* of f.

Remark: we have proven the following essential fact:

> A function $f : S \to T$ is a bijection if and only if it has a *two-sided inverse function* $f^{-1} : T \to S$.

The lecture proved the $\implies$ direction, and the above exercise proved the $\impliedby$ direction.

2. Let X, Y be sets, $f : X \to Y$ a function, and let A, B be subsets of X.

   (a) Is it always true that $f(A \cup B) = f(A) \cup f(B)$? Prove, or give a counterexample.

   (b) Is it always true that $f(A \cap B) = f(A) \cap f(B)$? Prove, or give a counterexample.

3. Let $m \neq 0$ and b be real numbers and consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = mx + b$. Prove that f is a bijection in two ways:

   (a) By showing f is injective and surjective directly.

   (b) By finding its inverse f^{-1}, and show it is an inverse by demonstrating that

   $$f^{-1}(f(x)) = x.$$

4. Let $\gamma, \rho \in \mathbb{R}$ be real numbers such that $\gamma \cdot \rho \neq 1$. Let $\mathbb{R} - \{\gamma\}$ and $\mathbb{R} - \{-\rho\}$ be the set of all real numbers $\mathbb{R}$ except for γ and $-\rho$, respectively. Consider the function $f : \mathbb{R} - \{-\rho\} \to \mathbb{R} - \{\gamma\}$ given by

   $$f(x) = \frac{\gamma x + 1}{x + \rho}.$$

   Show that f is a bijection.

5. Let S, T, and R be sets, and let $f : S \to T$ and $g : T \to R$ be functions. Show that if $g \circ f$ is bijective, then f is injective and g is surjective. Give an example to show that g and f need not be bijective.

6. Whenever a function $f : S \to T$ is not given explicitly on elements of its domain, we must check that it is *well-defined*, i.e. each element $s \in S$ takes on only a single value of T (under f). For instance, if f is defined on sets S_1 and S_2 such that $S = S_1 \cup S_2$, then checking that f is well-defined amounts to checking that the two definitions of f agree on the intersection $S_1 \cap S_2$ (remark: this is not the only important situation where well-definedness is crucial to check).

   (a) Why is the function $f : \mathbb{R} \to \mathbb{R}$ piecewise defined by

   $$f(x) = \begin{cases} x & x \geq 0 \\ -x & x \leq 0,\ x \geq -5 \\ x + 10 & x \leq -5 \end{cases}$$

   well-defined? (here $\mathbb{R}$ is the real numbers)

   (b) Why is the function $f : \mathbb{Z} \to \mathbb{Q}$ piecewise defined by

   $$f(x) = \begin{cases} x & x \text{ is a non-zero multiple of } 2 \\ \frac{1}{x} & x \text{ is a non-zero multiple of } 3, \\ -x & \text{otherwise} \end{cases}$$

   not well-defined? (here $\mathbb{Z}, \mathbb{Q}$ are the integers and rational numbers, respectively)

4 Real Sets

1. Let $a, b \in \mathbb{Z}$. Show that $4 \mid a^2 - b^2$ if and only if a and b are of the same parity.
2. Let $a \in \mathbb{Z}$. Show that $3 \mid a$ if and only if $3 \mid a^2$.
3. Let $a, b \in \mathbb{R}$. Show that if $a+b$ is rational, then a is irrational or b is rational.

5 Complex Numbers

1. Similar to how we obtained the double-angle formulae in the notes, use the Euler equation to show the two angle-sum formulae hold:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha;$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

2. Let z be a complex number.
   (a) Show that $|z| = \mathrm{Re}(z)$ if and only if z is a non-negative real number.
   (b) Show that $(\bar{z})^2 = z^2$ if and only if z is purely real or purely imaginary (i.e., its real part is 0).
3. The modulus of a complex number is, in many ways, a generalization of the absolute value of a real number. Here, we give another property of the modulus that the absolute value of a real number already enjoys.
   If $z, w \in \mathbb{C}$, show that

$$|z \cdot w| = |z| \cdot |w|$$

   in the following two ways:
   - By using the Cartesian form $z = a+bi$ and $w = c+di$ for the complex numbers z and w.
   - By using the polar form $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$ for the complex numbers z and w.

6 Induction

1. Let $r \neq 1$ be a real number. Use mathematical induction to show that

$$\sum_{j=0}^{n} r^j = \frac{1-r^{n+1}}{1-r}.$$

2. Let $x > -1$. Use mathematical induction to prove that

$$(1+x)^n \geq 1+nx$$

   for all integers $n \geq 1$.
3. Prove that for any positive integer n,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

   using induction.

4. $(*)$ Prove the *remainder theorem* for polynomial division using induction: If

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is a polynomial with integer coefficients, b is an integer, then there exists a polynomial $q(x)$, of degree $n-1$ and integer coefficients, and an integer r, such that

$$p(x) = (x-b)q(x) + r.$$

5. $(*)$ Prove *Fermat's Little Theorem* with induction. Namely, prove that for any integer a and any prime number p, we have $a^p \equiv a \pmod{p}$. You may use basic facts from modular arithmetic.