Individual Round 2022-2023

Problem 1. Given any four digit number $X = \underline{ABCD}$, consider the quantity $Y(X) = 2\cdot\underline{AB}+\underline{CD}$. For example, if $X = 1234$, then $Y(X) = 2\cdot 12+34 = 58$. Find the sum of all natural numbers $n \leq 10000$ such that over all four digit numbers X, the number n divides X if and only if it also divides $Y(X)$.

Problem 2. A sink has a red faucet, a blue faucet, and a drain. The two faucets release water into the sink at constant but different rates when turned on, and the drain removes water from the sink at a constant rate when opened. It takes 5 minutes to fill the sink (from empty to full) when the drain is open and only the red faucet is on, it takes 10 minutes to fill the sink when the drain is open and only the blue faucet is on, and it takes 15 seconds to fill the sink when both faucets are on and the drain is closed. Suppose that the sink is currently one-thirds full of water, and the drain is opened. Rounded to the nearest integer, how many seconds will elapse before the sink is emptied (keeping the two faucets closed)?

Problem 3. One of the bases of a right triangular prism is a triangle XYZ with side lengths $XY = 13, YZ = 14, ZX = 15$. Suppose that a sphere may be positioned to touch each of the five faces of the prism at exactly one point. A plane parallel to the rectangular face of the prism containing $\overline{YZ}$ cuts the prism and the sphere, giving rise to a cross-section of area A for the prism and area 15π for the sphere. Find the sum of all possible values of A.

Problem 4. Albert, Brian, and Christine are hanging out by a magical tree. This tree gives each of them a stick, each of which have a non-negative real length. Say that Albert gets a branch of length x, Brian a branch of length y, and Christine a branch of length z, and the lengths follow the condition that $x+y+z = 2$.

Let m and n be the minimum and maximum possible values of $xy+yz+xz-xyz$, respectively. What is $m+n$?

Problem 5. Let $\mathscr{S} := MATHEMATICSMATHEMATICSMATHE\ldots$ be the sequence where 7 copies of the word $MATHEMATICS$ are concatenated together. How many ways are there to delete all but five letters of $\mathscr{S}$ such that the resulting subsequence is $CHMMC$?

Problem 6. Consider two sequences of integers a_n and b_n such that $a_1 = a_2 = 1,\ b_1 = b_2 = 1$ and that the following recursive relations are satisfied for integers $n > 2$:

$$a_n = a_{n-1}a_{n-2} - b_{n-1}b_{n-2},$$
$$b_n = b_{n-1}a_{n-2} + a_{n-1}b_{n-2}.$$

Determine the value of

$$\sum_{1\leq n\leq 2023, b_n\neq 0} \frac{a_n}{b_n}.$$

Problem 7. Suppose ABC is a triangle with circumcenter O. Let A' be the reflection of A across $\overline{BC}$. If $BC = 12$, $\angle BAC = 60^\circ$, and the perimeter of ABC is 30, then find $A'O$.

Problem 8. A class of 10 students wants to determine the class president by drawing slips of paper from a box. One of the students, Bob, puts a slip of paper with his name into the box. Each other student has a $\frac{1}{2}$ probability of putting a slip of paper with their own name into the box and a $\frac{1}{2}$ probability of not doing so. Later, one slip is randomly selected from the box. Given that Bob's slip is selected, find the expected number of slips of paper in the box before the slip is selected.

Problem 9. Let a and b be positive integers, $a > b$, such that $6! \cdot 11$ divides $x^a - x^b$ for all positive integers x. What is the minimum possible value of $a + b$?

Problem 10. Find the number of pairs of positive integers (m, n) such that $n < m \leq 100$ and the polynomial $x^m + x^n + 1$ has a root on the unit circle.

Problem 11. Let ABC be a triangle and let ω be the circle passing through A, B, C with center O. Lines l_A, l_B, l_C are drawn tangent to ω at A, B, C respectively. The intersections of these lines form a triangle XYZ where X is the intersection of l_B and l_C, Y is the intersection of l_C and l_A, and Z is the intersection of l_A and l_B. Let P be the intersection of lines $\overline{OX}$ and $\overline{YZ}$. Given $\angle ACB = \frac{3}{2}\angle ABC$ and $\frac{AC}{AB} = \frac{15}{16}$, find $\frac{ZP}{YP}$.

Problem 12. Compute the remainder when

$$\sum_{1 \leq a,k \leq 2021} a^k$$

is divided by 2022 (in the above summation a, k are integers).

Problem 13. Consider a 7×2 grid of squares, each of which is equally likely to be colored either red or blue. Madeline would like to visit every square on the grid exactly once, starting on one of the top two squares and ending on one of the bottom two squares. She can move between two squares if they are adjacent or diagonally adjacent. What is the probability that Madeline may visit the squares of the grid in this way such that the sequence of colors she visits is *alternating* (i.e., red, blue, red, ... or blue, red, blue, ...)?

Problem 14. Let ABC be a triangle with $AB = 8$, $BC = 10$, and $CA = 12$. Denote by Ω_A the A-excircle of ABC, and suppose that Ω_A is tangent to $\overline{AB}$ and $\overline{AC}$ at F and E, respectively. Line $l \neq \overline{BC}$ is tangent to Ω_A and passes through the midpoint of $\overline{BC}$. Let T be the intersection of $\overline{EF}$ and l. Compute the area of triangle ATB.

Problem 15. For any positive integer n, let D_n be the set of ordered pairs of positive integers (m, d) such that d divides n and $\gcd(m, n) = 1$, $1 \leq m \leq n$. For any positive integers a, b, let $r(a, b)$ be the non-negative remainder when a is divided by b. Denote by S_n the sum

$$S_n = \sum_{(m,d) \in D_n} r(m, d).$$

Determine the value of S_{396}.