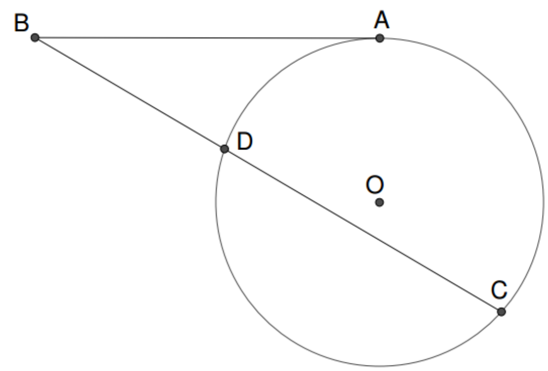
1. In the diagram below, point A lies on the circle centered at O. AB is tangent to circle O with $\overline{AB} = 6$. Point C is $\frac{2\pi}{3}$ radians away from point A on the circle, with BC intersecting circle O at point D. The length of BD is 3. Compute the radius of the circle.



Solution: The answer is $-\sqrt{3} + \sqrt{39}$.

First, using the tangent-secant power theorem, find $\overline{BC}$:

$$\overline{BC} = \frac{\overline{AB}^2}{\overline{BD}} = \frac{6^2}{3} = 12.$$

Then, knowing that $\angle AOC$ is $\frac{2\pi}{3}$, that $\angle OAB = \frac{\pi}{2}$ (tangent radius relationship), and that $\overline{AO} = \overline{CO}$ (both radii), it can be determined that $\angle BAC = \frac{2\pi}{3}$:

$$\angle OAC = \pi - \angle AOC - \angle OCA = \angle OCA = \frac{\pi - \frac{2\pi}{3}}{2} = \frac{\pi}{6},$$
$$\angle BAC = \angle OAB + \angle OAC = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}.$$

Next, find $\overline{AC}$ using the law of cosines:

$$\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 - 2\overline{AB}\,\overline{AC}\cos(\angle BAC)$$
$$144 = 36 + \overline{AC}^2 + 6\overline{AC}$$
$$\overline{AC} = \frac{-6 + \sqrt{6^2 + 4 \times 108}}{2}$$
$$\overline{AC} = \frac{-6 + 2\sqrt{117}}{2}$$
$$\overline{AC} = -3 + 3\sqrt{13}$$

Finally, bisect $\overline{AC}$ and use a 30-60-90 triangle to find $\overline{OA}$, the radius:

$$\overline{OA} = \frac{2}{\sqrt{3}} \cdot \frac{\overline{AC}}{2} = -\sqrt{3} + \sqrt{39}.$$

Hence, the answer is $-\sqrt{3} + \sqrt{39}$.

2. Suppose the roots of
$$x^4 - 3x^2 + 6x - 12 = 1$$
are α, β, γ, and δ. What is the value of
$$\frac{\alpha+\beta+\gamma}{\delta^2} + \frac{\alpha+\delta+\gamma}{\beta^2} + \frac{\alpha+\beta+\delta}{\gamma^2} + \frac{\delta+\beta+\gamma}{\alpha^2}\ ?$$

Solution: The answer is $-\frac{1}{2}$.

Because α, β, γ, and δ are the roots of the above equation, we know that their sum is the negative of the coefficient of the x^3 term, which is 0. Hence, we can simplify:
$$\begin{aligned}\frac{\alpha+\beta+\gamma}{\delta^2} + \frac{\alpha+\delta+\gamma}{\beta^2} + \frac{\alpha+\beta+\delta}{\gamma^2} + \frac{\delta+\beta+\gamma}{\alpha^2} &= \frac{0-\delta}{\delta^2} + \frac{0-\beta}{\beta^2} + \frac{0-\gamma}{\gamma^2} + \frac{0-\alpha}{\alpha^2} \\ &= -\frac{1}{\delta} - \frac{1}{\beta} - \frac{1}{\gamma} - \frac{1}{\alpha} \\ &= -\frac{-6}{-12} = -\frac{1}{2},\end{aligned}$$
as desired.

3. Bill plays a game in which he rolls two fair standard six-sided dice with sides labeled one through six. He wins if the number on one of the dice is three times the number on the other die. If Bill plays this game three times, compute the probability that he wins at least once.

Solution: The answer is $\frac{217}{729}$.

He has a $\frac{1}{9}$ chance of winning each game, and so his chance of winning at least once is $1 - (\frac{8}{9})^3$.

4. Let
$$\begin{aligned}A &= \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{9}, \\ B &= \frac{1}{2\cdot 3} + \frac{1}{2\cdot 5} + \frac{1}{2\cdot 9} + \frac{1}{3\cdot 5} + \frac{1}{3\cdot 9} + \frac{1}{5\cdot 9}, \\ C &= \frac{1}{2\cdot 3\cdot 5} + \frac{1}{2\cdot 3\cdot 9} + \frac{1}{2\cdot 5\cdot 9} + \frac{1}{3\cdot 5\cdot 9}.\end{aligned}$$
Compute the value of $A + B + C$.

Solution: The answer is $\frac{449}{720}$.

$1 + A + B + C + \frac{1}{270}$ factorizes to $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{9}\right)$ and can be shown to equal $\frac{8}{3}$. So the answer is $\frac{8}{3} - 1 - \frac{1}{270} = \frac{449}{720}$.