CMM 2026 Tiebreaker Round Solutions

Caltech Math Meet

January 24, 2026

1 Problem 1

Problem 1. Find the smallest positive integer N so that one can select 7 distinct positive divisors of N, no two of which divide each other.

Proposed by Justin Lee

Answer: $\boxed{420}$.

1.1 Solution

Observation: If two distinct factors of N are of the form $(p_1)^{a_1}(p_2)^{a_2}\dots(p_n)^{a_n}$ and $(p_1)^{b_1}(p_2)^{b_2}\dots(p_n)^{b_n}$ such that the sum of the a_i's equals the sum of the b_i's, then neither of these two factors divide each other. This is because each factor "dominates" the other on at least one prime, as in it contains more factors of this prime than the other does.

We can see from this that the optimal way to construct many factors, no two of which divide each other and yet are sufficiently small, is to choose a number S and have many distinct factors, each of which has S prime factors (including repeats).

Now we can do casework on how many distinct prime factors N has, in order to find the optimal value of N.

- **2 distinct prime factors:** Then the smallest construction of 7 factors, none of which divide each other, comes from $S = 6$: we have the 7 factors p^6, p^5q, p^4q^2, p^3q^3, p^2q^4, pq^5, and q^6, for distinct primes p, q. The LCM of all these factors is p^6q^6, so $p^6q^6 \mid N$ implying that the minimal value of N in this case is $3^6 \cdot 2^6$.

- **3 distinct prime factors:** If $S = 2$, then there are only 6 possible factors (the various ordered 3-tuples summing to 2), so $S \geq 3$. If $S = 3$, then the set of 7 factors each with sum of exponents 3 and such that their LCM is as small as possible, is the set $\{p^3, p^2q, p^2r, pq^2, pr^2, qr^2, pqr\}$ for distinct prime factors p, q, r. The LCM of these is $p^3q^2r^2$, so this must divide N, thus the minimal value of N in this case is $2^3 \cdot 3^2 \cdot 5^2 = 1800$.

- **4 distinct prime factors:** Clearly $S > 1$. If $S = 2$, then we have the set of 7 prime factors with sum of exponents 2: $\{p^2, pq, pr, ps, qr, qs, rs\}$ for distinct primes p, q, r, s. The LCM is p^2qrs, so the minimal value of N in this case is $2^2 \cdot 3 \cdot 5 \cdot 7 = 420$.

- **≥ 5 distinct prime factors:** The minimum positive integer with ≥ 5 distinct prime factors is > 420, so this case is irrelevant.

Over all the different cases, the minimum value of N is 420.

2 Problem 2

Problem 2. Two frogs, Vivian and Luke, start at $(0,0)$ and jump to $(3,3)$. Vivian can move either one unit right or one unit up on each move. Meanwhile, Luke can move one unit right, one unit up, or diagonally one unit right and up on each move. Each frog's path is drawn out on the grid. Compute the probability that all of the closed polygons formed by their paths do NOT form a total area of 1 or greater.

Proposed by Amudhan Gurumoorthy

Answer: $\boxed{\frac{4}{63}}$.

2.1 Solution

The only way possible that there is not a combined area 1 or greater is if Luke perfectly copies Vivian's path, except in places where Vivian switches from going up to right, to where Luke can choose one of these locations and jump directly across to cut through that place to form a polygon of area $\frac{1}{2}$. Thus, we just need to count all of these. We can do this by using two cases, where R means going right, U means going up, and D means going diagonally (up and right):

Case 1 : Luke and Vivian have the same exact path. In this case, we just count the number of paths Vivian has. Vivian must go $UUURRR$ in some order, so there are $\frac{6!}{3!3!} = 20$ many configurations.

Case 2 : Luke and Vivian differ in exactly one location. In that case, Luke's path must contain $UURRD$, and Vivian's path must be exactly the same as Luke's path, but replacing D with RU or UR. This results in $\frac{5!}{2!2!} \cdot 2 = 60$ possible cases.

Any other case results in a combined area of 1 or greater. Thus, there exist $20 + 60 = 80$ different cases.

Now, for the total number of cases. We know Vivian has 20 different paths. We can calculate Luke's possible paths by realizing that they must travel in a permutation $UUURRR$, $UUDRR$, $UDDR$, and DDD, which by counting have 20, 30, 12, and 1 different paths. Thus, Luke has $20 + 30 + 12 + 1 = 63$ different cases. Thus, the total number of configurations is $20 \cdot 63$, so the probability is $\frac{80}{20 \cdot 63} = \frac{4}{63}$.

3 Problem 3

Problem 3. In triangle $\triangle ABC$, point E is on line AC and F is on line AB, and points E' and F' are the reflections of E across the midpoint of AC and F across the midpoint of AB. Suppose that segment EF and line $E'F'$ intersect at point D on segment BC. If $\frac{DE}{DF} = \frac{3}{5}$ and $\frac{EE'}{FF'} = \frac{1}{3}$, what is $\frac{AE}{AF}$?

Proposed by Vivian Loh

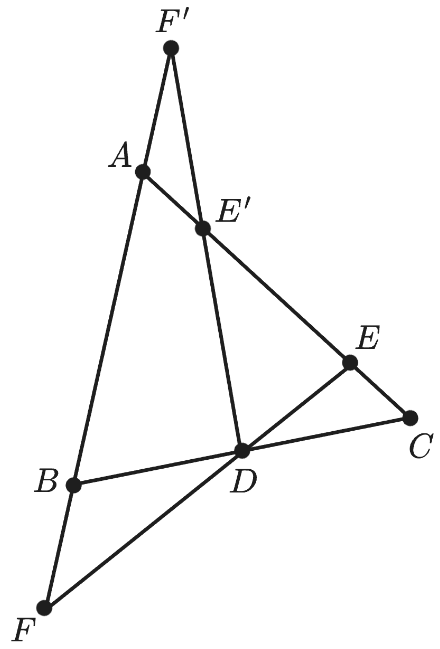
Answer: $\boxed{\frac{5}{9}}$.

3.1 Solution

Lemma 3.1
D must be the midpoint of side BC.

Proof. This can be proven using Menelaus's Theorem. Menelaus's Theorem states that if D' is the intersection of EF with BC, then $\frac{BD'}{D'C} = \frac{BF}{FA} \cdot \frac{AE}{EC}$, and if D'' is the intersection of $E'F'$ with BC then $\frac{CD''}{D''B} = \frac{CE'}{E'A} \cdot \frac{AF'}{F'B}$. These two products of fractions are equal, so $\frac{BD'}{D'C} = \frac{CD''}{D''B}$, which means that if $D' = D''$ then both must be the midpoint of BC. Intuitively, the intersections of EF and $E'F'$ with BC must be reflections over the midpoint of BC, which makes sense given all the symmetry about midpoints of sides in the diagram. □

Note that either E is contained in its side (AC) and F is not contained in its side (AB), or vice versa, so we technically have two cases. Let us evaluate the first case, in which E is contained in side AC. From the previous applications of Menelaus, we know that $\frac{AE}{EC} = \frac{AF}{FB}$. Thus there exists a real number x such that $EC : AE : AC = x : (1-x) : 1$, and $BF : AB : AF = x : (1-2x) : (1-x)$.



In order to use the condition $\frac{DE}{DF} = \frac{3}{5}$, we will apply Menelaus again with reference triangle $\triangle AEF$, and points D, B, C on lines EF, AF, AE respectively. We get $\frac{AB}{BF} \cdot \frac{FD}{DE} \cdot \frac{EC}{CA} = 1$, or $\frac{AB}{BF} \cdot \frac{EC}{CA} = \frac{3}{5}$. Writing these expressions in terms of x gives $\frac{1-2x}{x} \cdot \frac{x}{1} = 1 - 2x = \frac{3}{5}$, so $x = \frac{1}{5}$.

(Note that we can eliminate the other case, in which F is contained in side AB and E is not contained in side AC, because in this case defining x in the same way (as the ratio of BF to AB) would yield the unsolvable equation $1 - 2x = \frac{5}{3}$ instead, since the $\frac{3}{5}$ would be flipped to a $\frac{5}{3}$.)

Finally, we can use the $\frac{EE'}{FF'} = \frac{1}{3}$ condition to find the ratio $\frac{AE}{AF}$. Since we know $x = \frac{1}{5}$, points E' and E split side AC into segments in the ratio $1 : 3 : 1$, and points A and B split segment FF' into segments in the same ratio $1 : 3 : 1$. Thus, $\frac{AE}{EE'} = \frac{4}{3}$ and $\frac{FF'}{AF} = \frac{5}{4}$. So, the desired ratio $\frac{AE}{AF}$ is equal to

$$\frac{EE'}{FF'} \cdot \frac{AE}{EE'} \cdot \frac{FF'}{AF} = \frac{1}{3} \cdot \frac{4}{3} \cdot \frac{5}{4} = \boxed{\frac{5}{9}}.$$

4 Problem 4

Problem 4. Alice and Bob play a game. They start with a number $N = 2^p 3^q 5^r 7^s$ for nonnegative integers p, q, r, s. Then they take turns, with Alice going first, dividing the number N by an integer from 2 through 9 such that the quotient is a positive integer. The player who divides the number such that the quotient is 1 wins the game. Compute the smallest three-digit number N for which Bob will win the game.

Proposed by Amudhan Gurumoorthy

Answer: $\boxed{160}$.

4.1 Solution

Any combination that Alice can make to force the game to a number from $10 - 18$ will lead to a loss, because Bob must then force it to a single digit, and then Alice will just turn it straight to 1. This eliminates a lot of the earlier values that are possible, and simply counting out the rest of the cases results in 160 as an answer.