There are **12** problems, and you have **90** minutes. Answers are numbers that are not necessarily integers or positive. Answers should be reasonably simplified – answers which are integers should be fully written out, and **fractions should be simplified in lowest terms** meaning relatively prime numerator and denominator ($\frac{1}{2}$ instead of $\frac{2}{4}$). **Fractions should be used instead of decimals or mixed numbers.** Reduce radicals as much as possible, which includes **moving all powers outside of radicals** ($6\sqrt{5}$ instead of $2\sqrt{45}$), and **denominators should be rationalized** ($\frac{6\sqrt{5}}{5}$ instead of $\frac{6}{\sqrt{5}}$). Good luck and have fun!

1. The Fibonacci pirates are a motley gang of 7 pirates, Alpha, Beta, Gamma, Delta, Epsilon, Zeta, and Eta. They have a stash of $N > 0$ identical coins that they would like to divide up between themselves. They arrange themselves in a line in some order, and each pirate, when it is their turn to take their portion from the stash, does so with the following rules:
   - Alpha takes all of what remains in the stash.
   - Beta takes $\frac{1}{2}$ of what remains in the stash.
   - Gamma takes $\frac{1}{3}$ of what remains in the stash.
   - Delta takes $\frac{1}{5}$ of what remains in the stash.
   - Epsilon takes $\frac{1}{8}$ of what remains in the stash.
   - Zeta takes $\frac{1}{13}$ of what remains in the stash.
   - Eta takes $\frac{1}{21}$ of what remains in the stash.
   - Alpha, being the greediest of the pirates, is confined to go last.

   Suppose the pirates arrange themselves in some order so that each pirate receives an integer amount of coins. What is the fewest possible number of coins that Epsilon receives?

2. The arithmetic sequence $a_1, a_2, \dots$ with $a_1 \neq a_2$ satisfies that a_1, a_2, a_n and a_1, a_4, a_p are both geometric sequences. In this case, p can be represented as $cn + d$ for some integers c and d. Compute $c \cdot d$.

3. Ryan has a circular lock consisting of five equally spaced numbers, each a distinct digit from 1 to 5. Ryan does not know the correct code, so he inputs a random permutation of $1, \cdots, 5$ in the five slots. He can turn the lock by an arbitrary multiple of 72°, which rotates the positions of the numbers he inputted (without changing the value of the correct code). What is the probability that there is some turn of the lock (possibly 0°) for which at least two digits are in the correct positions?

4. For positive integers n, let $f(n)$ denote the number whose base-3 expansion coincides with the binary expansion of n. Compute the number of positive integers less than or equal to 2026 that can be written in the form $f(a+b) - f(a) - f(b)$ for some positive integers a and b.

5. Find the number of ways to fill each cell of a 10×10 grid with a 0 or a 1 such that every 2×2 subgrid contains exactly two 0s and exactly two 1s.

6. Two strips of paper with widths 10 and 14 overlap on a region that is a parallelogram with area 168. A point P lies inside this parallelogram such that the feet of the altitudes from P to the four sides of the parallelogram all lie on a circle with area 50π. The four possible points P which satisfy this form another parallelogram. What is its area?

7. Let $a_0, a_1, \cdots, a_{10}$ be a strictly increasing sequence of positive integers such that $a_0 = 0$, $a_{10} = 67$, and the difference between every pair of consecutive terms is either 6 or 7. How many such sequences exist for which $\gcd(a_i, a_{i+1}) = 1$ for each $i \in \{1, 2, 3, \cdots, 8, 9\}$?

8. Triangle ABC has area 1. Points D and E are on the opposite side of line BC from A such that triangles BDC and CBE are both similar to triangle ABC, and have areas 2 and 4 respectively. The perpendicular line from B to CD intersects the perpendicular line from C to BE at point F. The length of AF can be expressed as $\frac{a}{b^{3/4}}$, for relatively prime positive integers a and b. Find $a + b$.

9. Find the smallest positive integer a for which there exist positive integers b and c such that

$$\frac{a}{b^2} + \frac{1}{c} = \frac{1}{2026}.$$

10. A cube is suspended in the air with its eight vertices 1, 2, 3, 4, 5, 6, 7, and 8 units off of the ground. What is its side length?

11. Suppose the polynomial $x^4 + ax^3 + 69x^2 + bx + 100$ has real coefficients and roots z_1, z_2, z_3, and z_4, none of which are real. The possible values of $|z_1| + |z_2| + |z_3| + |z_4|$ form an open interval (m, M). Compute m.

12. Three people are each initially given the number 0. Every hour, they each simultaneously flip a fair coin: for each person, if their coin lands as heads, they increase their number by one; otherwise, they don't change their number. What is the expected number of hours until the first occurrence of the three numbers all being distinct?