CHMMC Team Round Problems

November 22, 2015

Problem 0.1. 3 *players take turns drawing lines that connect vertices of a regular* n*-gon. No player may draw a line that intersects another line at a point other than a vertex of the* n*-gon. The last player able to draw a line wins. For how many* n *in the range* $4 \leq n \leq 100$ *does the first player have a winning strategy?*

Solution 1. $\boxed{33 \textit{ or } 32}$.

The intended interpretation of this problem was that legal plays in the game consist only of diagonals.

The number of lines drawn during the game is always $n-3$*. We prove this by induction on* n*.*

Suppose inductively that every instance of the game on an $n-gon$ *vertices always has* $n-3$ *lines drawn. Consider an instance of the game on* n *vertices. There must be some line drawn between two vertices with just one vertex between them. (If not, then the game isn't over, as there is still room to draw such a line.) This line partitions the* $(n+1)-gon$ *into an* n*-gon and a triangle. We can consider the rest of the game to take place on the* n*-gon; it will therefore have exactly* $n-3$ *moves. Therefore the game on the* $(n+1)$*-gon takes exactly* $n-2$ *moves.*

The first player wins when $n \equiv 1 \pmod 3$*. There are* 33 *such* n *in this range.*

An alternative interpretation of the problem is that edges of the polygon are also legal moves. In this case, there are $2n-3$ *legal moves, so the first player wins when* $n \equiv 2 \pmod 3$*, yielding* 32 *possible* n *over the range. This answer was also accepted on competition day.*

Problem 0.2. *You have 4 game pieces, and you play a game against an intelligent opponent who has 6. The rules go as follows: you distribute your pieces among two points* a *and* b*, and your opponent simultaneously does as well (so neither player sees what the other is doing). You win the round if you have more pieces than them on either* a *or* b*, and you lose the round if you only draw or have fewer pieces on both. You play the optimal strategy, assuming your opponent will play with the strategy that beats your strategy most frequently. What proportion of the time will you win?*

Solution 2. $\boxed{\frac{2}{3}}$.

If you number your plays as $p_i = (i, 4-i)$ *and theirs as* $q_j = (j, 6-j)$*, you notice that* p_i *beats the 4 enemy plays with either* $j < i$ *or* $6-j < 4-i$*. Clearly a symmetric strategy will be best (where play* p_i *is equally likely as* p_{4-i}*), so let* a, b, c *be the probabilities of plays* $(0,4), (1,3)$*, and* $(2,2)$ *respectively. Normalizing demands that* $2a+2b+c = 1$*. This beats states* $(0,6), (1,5), (2,4), (3,3)$ *with probabilities* $a+2b+c, a+b+c, a+b, 2a$ *respectively. This suggests the play* p_2 *is useless, since the opponent will never win more often with* $(0,6)$ *or* $(1,5)$ *than*

(2, 4), and therefore never play them. Maximizing $\min(2a, a+b)$ *gives* $a = b$*, so your optimal strategy will have* $a = b$*. Normalizing gives* $a = b = \frac{1}{3}$*, so you win* $\boxed{\frac{2}{3}}$ *of the time.*

Problem 0.3. *A trio of lousy salespeople charge increasing prices on tomatoes as you buy more. The first charges you* x_1 *dollars for the* x_1*th tomato you buy from him, the second charges* x_2^2 *dollars for the* x_2*th tomato, and the third charges* x_3^3 *dollars for the* x_3*th tomato. If you want to buy* 100 *tomatoes for as cheap as possible, how many should you buy from the first salesperson?*

Solution 3. $\boxed{87}$.

You should always buy the cheapest tomato, so we want to keep x_1, x_2^2, x_3^3 *roughly equal; our total tomatoes purchased should then be approximately* $x_1 + \sqrt{x_2} + \sqrt[3]{x_3}$*. Since* x_i *are integers, we need to find the* x_1 *such that*

$$\lfloor x_1 \rfloor + \lfloor \sqrt{x_1} \rfloor + \lfloor \sqrt[3]{x_1} \rfloor = 100$$

Trying a few values and adjusting gives $\boxed{x_1 = 87}, x_2 = 9, x_3 = 4$.

Problem 0.4. *Let* $P(x) = x^{16} - x^{15} + \cdots - x + 1$*, and let* p *be a prime such that* $p - 1$ *is divisible by* 34 *(*$p = 103$ *is an example). How many integers* a *between* 1 *and* $p - 1$ *inclusive satisfy the property that* $P(a)$ *is divisible by* p*?*

Solution 4. $\boxed{16}$.

$P(a)$ *is divisible by* p *if and only if* $P(a) \equiv 0 \pmod{p}$*. There are* p *residue classes modulo* p*, and each of them is represented by exactly one integer between* 0 *and* $p - 1$ *inclusive.* $P(0) = 1$*, so* $P(0) \not\equiv 0 \pmod{p}$*, so we need only count the residue classes modulo* p *for which* $P(a) \equiv 0 \pmod{p}$.

$(x+1)P(x) = x^{17} + 1$*, so if* a *is a solution, and* $P(a) \equiv 0 \pmod{p}$*, then* $a^{17} + 1 \equiv 0 \pmod{p}$*. If* $a^{17} + 1 \equiv 0 \pmod{p}$ *and* $a \not\equiv -1 \pmod{p}$*, then* $(a+1)P(a) \equiv 0 \pmod{p}$*, and* $P(a) \equiv 0 \pmod{p}$*, so* a *is one of our solutions. If* $a \equiv -1 \pmod{p}$*, then* $P(a) \equiv P(-1) \equiv 17 \not\equiv 0 \pmod{p}$*. So our solutions are exactly those* a *for which* $a \not\equiv -1 \pmod{p}$ *but* $a^{17} \equiv -1 \pmod{p}$*. As* $a^{17} \equiv -1 \pmod{p}$*, squaring gives us* $a^{34} \equiv 1 \pmod{p}$*, so we are looking for residues that have order dividing 34. As* $a \not\equiv -1 \pmod{p}$*,* a *does not have order* 2*, and as* $a^{17} \not\equiv 1 \pmod{p}$*,* a *does not have order* 1 *or* 17*. Thus our solutions are those integers with order exactly* 34*.*

It is a fact from elementary number theory and group theory that there exists some integer g *such that the exponents* $g^1, g^2, \ldots, g^{p-1}$ *are equivalent modulo* p *to* $1, 2, \ldots, p-1$ *in some order. By Fermat's Little Theorem, for a given exponent* m*,* $(g^n)^m \equiv 1 \pmod{p}$ *if and only if* $mn \equiv 0 \pmod{p-1}$*, and thus* $p - 1$ *divides* mn*. Thus* g^n *has order* 34 *if and only if* $p - 1$ *divides* $34n$ *but not* $2n$ *or* $17n$*. Thus* n *is a multiple of* $\frac{p-1}{34}$*, so* $n = k\frac{p-1}{34}$ *for some* $1 \leq k \leq 34$*. If* k *is even, then* $\frac{p-1}{17}$ *divides* n*, so* $17n$ *divides* $p - 1$*, so* g^n *is not a solution. Similarly, if* k *is a multiple of* 17*, then* $\frac{p-1}{2}$ *divides* $p - 1$*, and* g^n *is not a solution. Therefore each of our solutions corresponds to a number* k *with* $1 \leq k \leq 34$*, and* k *is not a multiple of* 2 *or* 17*. There are* $34 - 17 - 1 = 16$ *such numbers, so our answer is* $\boxed{16}$.

Problem 0.5. *Felix is playing a card-flipping game.* n *face-down cards are randomly colored, each with equal probability of being black or red. Felix starts at the* 1*st card. When Felix is at the* k*th card, he guesses its color and then flips it over. For* $k < n$*, if he guesses correctly, he moves onto the* $(k+1)$*-th card. If he guesses incorrectly, he gains* k *penalty points, the cards are replaced with newly randomized face-down cards, and he moves back to card* 1 *to continue guessing. If Felix guesses the* n^{th} *card correctly, the game ends.*

What is the expected number of penalty points Felix earns by the end of the game?

Solution 5. $\boxed{2^{n+1} - n - 2}$.

The expected number of times Felix starts from the beginning is is 2^n. *The expected number of points on each of these runs is*

$$\sum_{j=1}^{n} \frac{j}{2^j} = \sum_{j=1}^{n} \sum_{i=j}^{n} \frac{1}{2^i} = \sum_{j=1}^{n} \left(\frac{1}{2^{j-1}} - \frac{1}{2^n} \right) = 2 - \frac{1}{2^{n-1}} - \frac{n}{2^n} = \frac{2^n - n - 2}{2^n}$$

Problem 0.6. *The icosahedron is a convex, regular polyhedron consisting of* 20 *equilateral triangle for faces. A particular icosahedron given to you has labels on each of its vertices, edges, and faces. Each minute, you uniformly at random pick one of the labels on the icosahedron. If the label is on a vertex, you remove it. If the label is on an edge, you delete the label on the edge along with any labels still on the vertices of that edge. If the label is on a face, you delete the label on the face along with any labels on the edges and vertices which make up that face. What is the expected number of minutes that pass before you have removed all labels from the icosahedron?*

Solution 6. $\boxed{\frac{342}{11}}$.

Let V, E, F *denote the number of vertices, edges, and faces respectively, of the icosahedron. Then we are given that* $F = 20$. *From double counting and the fact that each face has* 3 *edges, we see that*

$$2E = 3F \implies E = 30.$$

Then by Euler's formula for polyhedra, we have

$$V - E + F = 2 \implies V = 12.$$

Since the icosahedron is a regular polyhedron, there is a positive integer ℓ *such that exactly* ℓ *faces meet at each vertex. By a triple counting argument, we find that*

$$\ell V = 3F \implies \ell = 5.$$

Similarly, 5 *edges meet at each vertex.*
Now that we have extracted the necessary information about the icosahedron, we may solve the actual problem. First, note that the expected number of minutes that pass before removing all labels is the same as the expected number of labels that are removed. By linearity of expectation, the expected number of labels removed is just the sum of the probabilities that any single label is picked (note that this is different from being removed), taken over all labels.
Each of the 20 *face labels has probability* 1 *of being picked at some point, since the only way to remove a face label is to pick it.*
An edge label on the other hand, is picked if and only if at the time of it being picked it is selected before any of the face labels containing that edge has been picked. Since there are two faces containing any edge, and each of these three labels are equally likely to be picked at any given time, each of the 30 *edges has probability* $1/3$ *of being chosen.*
Finally, a vertex label is picked if and only if it is chosen before the labels on any of the faces or edges containing that vertex have been picked. As seen above, there are 5 *edges and* 5 *faces containing any given vertex. Any one of these* 11 *labels has equal probability of being picked first. Thus, each of the* 12 *vertices has probability* $1/11$ *of being picked.*

Then the expected number of labels that are removed is just

$$20 \cdot 1 + 30 \cdot \frac{1}{3} + 12 \cdot \frac{1}{11} = \frac{342}{11}.$$

Problem 0.7. *Let I be the incenter and let Γ be the incircle of $\triangle ABC$, and let $P = \Gamma \cap BC$. Let Q denote the intersection of Γ and the line passing through P parallel to AI. Let ℓ be the tangent line to Γ at Q and let $\ell \cap AB = S, \ell \cap AC = R$. If $AB = 7, BC = 6, AC = 5$, what is RS?*

Solution 7. $\boxed{1}$.

Let $AB = c, BC = a, AC = b$.

Let $AI \cap SR = T$ and $AI \cap BC = U$. Let $\angle ITQ = \theta$. Since QT is tangent to Γ, $\angle QIT = 90 - \theta$. Because AI is parallel to PQ, $\angle IQP = \angle QIT = 90 - theta$. QIP is isosceles, so $\angle QIP = 2\theta$. Thus AI and IP make an angle of $90 - \theta$ because the total angle along AI is 180 degrees.

Since $IP \perp toBC$, this implies $\angle IUP = \theta$. Thus by AA similarity $ATR \sim AUB$ which means $\angle ARS = \angle B$, so $ARS \sim ABC$.

Because of this similarity, we can dilate ASR so as to take its incenter to Γ because AI is also the A-angle bisector in ASR. Let the image of the dilation be to $AS'R'$ which is congruent to ABC since it is similar to ABC and has the same size incircle. The dilation factor is can be calculated as the ratio of the altitudes of $AS'R'$ and ASR which $\frac{h_a}{h_a - 2r}$ because $SR || S'R'$ and the distance between them is $2r$ because SR which is tangent to Γ is mapped to another line tangent to Γ and Γ has diameter $2r$, where h_a, r are the A-altitude and inradius of ABC respectively. Since $2rs = ah_a$, this dilation factor can be rewritten as $(1 - \frac{2r}{h_a}) = 1 - \frac{a}{s}$. Thus $|SR|\frac{s-a}{s} = a \Rightarrow |SR| = \frac{as - a^2}{s} = \frac{a(b+c-a)}{2(a+b+c)} = \frac{6(5+7-6)}{2(6+5+7)} = \boxed{1}$.

Problem 0.8.

$$\text{Let } f(n) = \sum_{d=1}^{n} \left\lfloor \frac{n}{d} \right\rfloor \text{ and } g(n) = f(n) - f(n-1).$$

For how many n from 1 to 100 inclusive is $g(n)$ even?

Solution 8. $\boxed{90}$.

Consider the following function:

$$h(n, d) = \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{n-1}{d} \right\rfloor.$$

Notice that $h(n)$ is nonzero if and only if $d|n$ in which case $h(n) = 1$. Furthermore, we have

$$g(n) = \sum_{d=1}^{n} h(n, d).$$

Therefore $g(n)$ just counts the total number of divisors of n. Only perfect squares have an odd number of divisors, and there are 10 perfect squares in the range $[1, 100]$.

Problem 0.9. *Let T be a 2015×2015 array containing the integers $1, 2, 3, \ldots, 2015^2$ satisfying the property that $T_{i,a} > T_{i,b}$ for all $a > b$ and $T_{c,j} > T_{d,j}$ for all $c > d$ where $1 \le a, b, c, d \le 2015$ and $T_{i,j}$ represents the entry in the ith row and jth column of T. How many possible values are there for the entry at $T_{5,5}$?*

Solution 9. $\boxed{16081}$.

$T_{5,5}$ must be greater than all the entries in the top left 5×5 subarray, which consists of $5^2 - 1$ extra entries, so $T_{5,5} \geq 5^2$. Similarly, $T_{5,5}$ cannot be greater than any of the entries in the bottom right $(2015-5+1) \times (2015-5+1)$ subarray, so $T_{5,5} \leq 2015^2 - (2015-5+1)^2 + 1$. Thus the number of possibilities for $T_{5,5}$ is $(2015^2 - (2015-5+1)^2 + 1) - 5^2 + 1 = (2 \cdot 2015 + 2) \cdot 5 - 2 \cdot 5^2 - (2 \cdot 2015 - 1) = 16081$.

Problem 0.10. *Let $\mathcal{P}$ be the parabola in the plane determined by the equation $y = x^2$. Suppose a circle $\mathcal{C}$ in the plane intersects $\mathcal{P}$ at four distinct points. If three of these points are $(-28, 784), (-2, 4)$, and $(13, 169)$, find the sum of the distances from the focus of $\mathcal{P}$ to all four of the intersection points.*

Solution 10. $\boxed{1247}$.

Let the intersection points of $\mathcal{C}$ and $\mathcal{P}$ be $\left(a, a^2\right), \left(b, b^2\right), \left(c, c^2\right)$, and $\left(d, d^2\right)$. First, since $\mathcal{P}$ is a parabola, by definition the distance from its focus to any one of the points $\left(x, x^2\right)$ on the parabola is just

$$x^2 + \frac{1}{4},$$

since the directrix of $\mathcal{P}$ is the line with equation $y = -1/4$.
Suppose the circle $\mathcal{C}$ is centered at the point (h, k) and radius r, so that it is modeled by the equation

$$(y - k)^2 + (x - h)^2 = r^2.$$

Since the equation of $\mathcal{P}$ is $y = x^2$, we further find that the reals a, b, c, d must be roots of the quartic

$$\left(x^2 - k\right)^2 + (x - h)^2 - r^2 = 0.$$

The coefficient of x^3 in this quartic is zero, so that by vieta's formulas we know that

$$a + b + c + d = 0 \iff d = -(a + b + c).$$

Thus, if we are given the x coordinates of three of the intersection points, a, b, c, as we have been, the desired answer is just

$$\begin{aligned} &a^2 + 1/4 + b^2 + 1/4 + c^2 + 1/4 + d^2 + 1/4 \\ &= a^2 + b^2 + c^2 + (a + b + c)^2 + 1 \\ &= (a + b)^2 + (b + c)^2 + (c + a)^2 + 1 \\ &= 900 + 121 + 225 + 1 = \boxed{1247}. \ \blacksquare \end{aligned}$$