CHMMC 2023 Integration Bee Final Round Solutions

Problem 1. $\int_1^e \frac{\cos(\ln x)}{x}\,\mathrm{dx}$

Proposed by Brian Yang

Solution: $\boxed{\sin(1)}$

We note that $(\ln(x))' = 1/x$, which implicates the u-substitution involving $u = \ln(x)$ and $\mathrm{du} = 1/x\,\mathrm{dx}$. Therefore, we have the following equivalent integral.

$$\int_1^e \frac{\cos(\ln(x))}{x}\,\mathrm{dx} = \int_0^1 \cos(u)\,\mathrm{du}$$

We can integrate this regularly to get the following answer.

$$\int_0^1 \cos(u)\,\mathrm{du} = \sin(u)\Big|_0^1 = \sin(1) - \sin(0) = \boxed{\sin(1)}$$

Problem 2. $\int \frac{\cos(x)\csc(x)}{\cot(x)}\,\mathrm{dx}$

Proposed by Jeck Lim

Solution: $\boxed{x + C}$

We expand each of the components in terms of $\cos(x)$ and $\sin(x)$ as follows.

$$\int \frac{\cos(x)\csc(x)}{\cot(x)}\,\mathrm{dx} = \int \frac{\cos(x)\cdot\left(\frac{1}{\sin(x)}\right)}{\frac{\cos(x)}{\sin(x)}}\,\mathrm{dx} = \int \frac{\frac{\cos(x)}{\sin(x)}}{\frac{\cos(x)}{\sin(x)}}\,\mathrm{dx} = \int 1\,\mathrm{dx} = \boxed{x + C}$$

Problem 3. $\int \sin(\cos(\cos x))\cdot\sin(\cos(x))\cdot\sin(x)\,\mathrm{dx}$

Proposed by Jeck Lim

Solution: $\boxed{-\cos(\cos(\cos(x))) + C}$

We note the nested trigonometric expressions, so we consider $u = \cos(\cos(x))$, as we have no neat way to integrate this as an argument of sin. Thus, we have the following expression of du.

$$\mathrm{du} = -\sin(\cos(x))\cdot -\sin(x)\,\mathrm{dx} = \sin(\cos(x))\cdot\sin(x)\,\mathrm{dx}$$

These two components match up with the last two components of the integral, which produces the equivalent integral as follows.

$$\int \sin(\cos(\cos x))\cdot[\sin(\cos(x))\cdot\sin(x)\,\mathrm{dx}] = \int \sin(u)\,\mathrm{du} = -\cos(u) + C$$

Substituting u, we have the final anti-derivative.

$$-\cos(u) + C = \boxed{-\cos(\cos(\cos(x))) + C}$$

Problem 4. $\int_{-1}^{1} x^2 \cdot \sqrt[3]{x^3+1}\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\frac{\sqrt[3]{2}}{2}}$

We note that $3x^2 = (x^3+1)'$, so we can utilize a u-substitution.

$$\int_{-1}^{1} x^2 \cdot \sqrt[3]{x^3+1}\ \mathrm{dx} \underbrace{\Longrightarrow}_{u=x^3+1} \int_0^2 \left(\frac{du}{3}\right) \cdot \sqrt[3]{u} = \frac{1}{3} \cdot \int_0^2 u^{\frac{1}{3}}\ \mathrm{du}$$

Evaluating this integral, we have the following.

$$\frac{1}{3} \cdot \int_0^2 u^{\frac{1}{3}}\ \mathrm{du} = \frac{1}{3} \cdot \frac{u^{\frac{1}{3}+1}}{\frac{1}{3}+1}\Bigg|_0^2 = \frac{1}{3} \cdot \frac{3u^{\frac{4}{3}}}{4}\Bigg|_0^2 = \frac{2^{\frac{4}{3}}}{4} - \frac{0^{\frac{4}{3}}}{4} = \frac{2\sqrt[3]{2}}{4} = \boxed{\frac{\sqrt[3]{2}}{2}}$$

Problem 5. $\int \sqrt{\sec(x)} \cdot \tan(x)\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{2\sqrt{\sec(x)} + C}$

We note that $(\sec(x))' = \sec(x) \cdot \tan(x)$, so we try to reform the integral into this form.

$$\int \sqrt{\sec(x)} \cdot \tan(x) \cdot \frac{\sqrt{\sec(x)}}{\sqrt{\sec(x)}}\ \mathrm{dx} = \int \frac{\sec(x) \cdot \tan(x)}{\sqrt{\sec(x)}}\ \mathrm{dx}$$

From here, we can utilize a u-substitution of $u = \sec(x)$ as follows.

$$\int \frac{\sec(x) \cdot \tan(x)}{\sqrt{\sec(x)}}\ \mathrm{dx} \underbrace{\Longrightarrow}_{u=\sec(x)} \int \frac{\mathrm{du}}{\sqrt{u}} = \int u^{\frac{-1}{2}}\ \mathrm{du}$$

This is a simple use of the power rule for integrals.

$$\int u^{\frac{-1}{2}}\ \mathrm{du} = \frac{u^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} + C = 2u^{\frac{1}{2}} + C = \boxed{2\sqrt{\sec(x)} + C}$$

Problem 6. $\int e^{e^x + x}\ \mathrm{dx}$

Proposed by Jeck Lim

Solution: $\boxed{e^{e^x} + C}$

Expanding the integral, we have the following.

$$\int e^{e^x + x}\ \mathrm{dx} = \int e^{e^x} \cdot e^x\ \mathrm{dx}$$

We consider the u-substitution of $u = e^x$, which also implies that du $= e^x$ dx. Therefore, we have the following.

$$\int e^{e^x} \cdot e^x \text{ dx} = \int e^u \text{ du} = e^u + C = \boxed{e^{e^x} + C}$$

Problem 7. $\int_1^{10} e^{\ln x} + \ln e^x \text{ dx}$

Proposed by Jeck Lim

Solution: $\boxed{99}$

Both of the components can be simplified as follows.

$$e^{\ln(x)} = x \qquad\qquad \ln(e^x) = x$$

Thus, the integral becomes the following.

$$\int_1^{10} e^{\ln x} + \ln e^x \text{ dx} = \int_1^{10} (x + x) \text{ dx} = \int_1^{10} 2x \text{ dx}$$

Evaluating this integral, we have the following.

$$\int_1^{10} 2x \text{ dx} = x^2 \Big|_1^{10} = 10^2 - 1^2 = 100 - 1 = \boxed{99}$$

Problem 8. $\int_{-2023}^{2023} \frac{\sin(x)}{x^2 + 1} \text{ dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{0}$

Note that by the property of an odd function $o(x)$, we have the following if integrating across symmetric bounds.

$$\int_{-a}^{a} o(x) \text{ dx} = 0$$

We can clearly verify that the integrand is odd since $\sin(x)$ is odd and $(x^2 + 1)$ is even. This also means that their quotient will also be odd. Therefore, we can use the aforementioned property as follows since we have symmetric bounds.

$$\int_{-2023}^{2023} \frac{\sin(x)}{x^2 + 1} \text{ dx} = \boxed{0}$$

Problem 9. $\int_0^1 \frac{x}{x^4 + 1} \text{ dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\frac{\pi}{8}}$

We consider re-writing $x^4 = (x^2)^2$, which gives us appropriate motivation to consider a u-substitution of $u = x^2$. This implies that $\mathrm{du} = 2x\ \mathrm{dx}$ and the following equivalent integral.

$$\int_0^1 \frac{x}{(x^2)^2+1}\ \mathrm{dx} = \int_0^1 \frac{\left(\frac{\mathrm{du}}{2}\right)}{u^2+1}$$

This is easily recognizable as the arc-tangent derivative, which allows us to simplify it as follows.

$$\frac{1}{2} \cdot \int_0^1 \frac{\mathrm{du}}{u^2+1} = \frac{\arctan(u)}{2}\bigg|_0^1 = \frac{\arctan(1)}{2} - \frac{\arctan(0)}{2} = \frac{\frac{\pi}{4}}{2} - 0 = \boxed{\frac{\pi}{8}}$$

Problem 10. $\displaystyle\int \frac{\mathrm{dx}}{x \cdot \sqrt{1-(\ln(x))^2}}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\arcsin(\ln(x)) + C}$

We note that $(\ln(x))' = 1/x$, which implicates the u-substitution involving $u = \ln(x)$ and $\mathrm{du} = 1/x\ \mathrm{dx}$. Therefore, we have the following equivalent integral.

$$\int \frac{\mathrm{dx}}{x \cdot \sqrt{1-(\ln(x))^2}} = \int \frac{\mathrm{du}}{\sqrt{1-u^2}}$$

This integral is easily recognizable as the derivative of $\arcsin(u)$, which gives us the following anti-derivative upon substituting back into u.

$$\int \frac{\mathrm{du}}{\sqrt{1-u^2}} = \arcsin(u) + C = \boxed{\arcsin(\ln(x)) + C}$$

Problem 11. $\displaystyle\int \frac{\mathrm{dx}}{e^x + e^{-x}}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\arctan(e^x) + C}$

We can multiply the numerator and denominator by e^x as follows.

$$\int \frac{\mathrm{dx}}{e^x + e^{-x}} \cdot \frac{e^x}{e^x} = \int \frac{e^x\ \mathrm{dx}}{e^{2x}+1}$$

We can now consider the u-substitution of $u = e^x$, which also implies that $\mathrm{du} = e^x\ \mathrm{dx}$. Thus, the equivalent integral is as follows which can be recognized as the derivative of arc-tangent.

$$\int \frac{e^x\ \mathrm{dx}}{e^{2x}+1} = \int \frac{\mathrm{du}}{u^2+1} = \arctan(u) + C = \boxed{\arctan(e^x) + C}$$

Problem 12. $\displaystyle\int_0^1 (x-1)^2(x+1)^2(x^2+1)^2(x^4+1)^2\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\frac{128}{153}}$

We can combine all of the components methodically as follows.

$$(x-1)^2 \cdot (x+1)^2 = ((x-1)(x+1))^2 = (x^2-1)^2$$
$$(x^2-1)^2 \cdot (x^2+1)^2 = ((x^2+1)(x^2-1))^2 = (x^4-1)^2$$
$$(x^4-1)^2 \cdot (x^4+1)^2 = ((x^4-1)(x^4+1))^2 = (x^8-1)^2$$

Therefore, the integral becomes the following.

$$\int_0^1 (x^8-1)^2 \text{ dx} = \int_0^1 x^{16} - 2x^8 + 1 \text{ dx} = \left(\frac{x^{17}}{17} - \frac{2x^9}{9} + x\right)\Bigg|_0^1$$

Evaluating, we have the following.

$$\left(\frac{x^{17}}{17} - \frac{2x^9}{9} + x\right)\Bigg|_0^1 = \left(\frac{1}{17} - \frac{2}{9} + 1\right) = \frac{9}{153} - \frac{34}{153} + \frac{153}{153} = \boxed{\frac{128}{153}}$$

Problem 13. $\int_{\frac{1}{3}}^{3} \ln(e^{\lfloor \frac{1}{x} \rfloor}) \text{ dx}$

Proposed by Brian Yang

Solution: $\boxed{\frac{5}{6}}$

We can simplify the integral using logarithm properties as follows.

$$\int_{\frac{1}{3}}^{3} \ln(e^{\lfloor \frac{1}{x} \rfloor}) \text{ dx} = \int_{\frac{1}{3}}^{3} \left\lfloor \frac{1}{x} \right\rfloor \text{ dx}$$

We note that for any $x > 1$, this implies that $1/x < 1$ and that the floor of this value will be 0. Therefore, we can split the integral as follows, noting that the latter integral will evaluate to 0.

$$\int_{\frac{1}{3}}^{3} \left\lfloor \frac{1}{x} \right\rfloor \text{ dx} = \int_{\frac{1}{3}}^{1} \left\lfloor \frac{1}{x} \right\rfloor \text{ dx} + \int_{1}^{3} \left\lfloor \frac{1}{x} \right\rfloor \text{ dx} = \int_{\frac{1}{3}}^{1} \left\lfloor \frac{1}{x} \right\rfloor \text{ dx}$$

We note that on the interval of $x \in [1/2, 1]$, the integrand will evaluate to 1. Additionally, for $x \in [1/3, 1/2]$, the integrand will evaluate to 2. Therefore, we have the following final answer.

$$\int_{\frac{1}{3}}^{1} \left\lfloor \frac{1}{x} \right\rfloor \text{ dx} = \int_{\frac{1}{3}}^{1/2} 2 \text{ dx} + \int_{1/2}^{1} 1 \text{ dx} = 2 \cdot \left(\frac{1}{2} - \frac{1}{3}\right) + 1 \cdot \left(1 - \frac{1}{2}\right) = \boxed{\frac{5}{6}}$$

Problem 14. $\int_{-\infty}^{\infty} \frac{xe^{-x^2}}{\ln(x^2+2)} \text{ dx}$

Proposed by Jeck Lim

Solution: $\boxed{0}$

Note that by the property of an odd function $o(x)$, we have the following if integrating across symmetric bounds.

$$\int_{-a}^{a} o(x)\ \mathrm{dx} = 0$$

We can clearly verify that the integrand is odd since xe^{-x^2} is odd and $\ln(x^2+2)$ is even. This also means that their quotient will also be odd. Therefore, we can use the aforementioned property as follows since we have symmetric bounds.

$$\int_{-\infty}^{\infty} \frac{xe^{-x^2}}{\ln(x^2+2)}\ \mathrm{dx} = \boxed{0}$$

Problem 15. $\displaystyle\int_0^3 \lceil x\rceil \cdot x^{\lceil x\rceil} - \lfloor x\rfloor\ \mathrm{dx}$

Proposed by Brian Yang

Solution: $\boxed{\dfrac{611}{12}}$

We can split the integral into the intervals, namely $[0,1]$, $[1,2]$, and $[2,3]$. On each of these intervals, we have that $\lceil x\rceil = \{1,2,3\}$ and $\lfloor x\rfloor = \{0,1,2\}$, respectively. We can calculate each of the integrals as follows.

$$\int_0^1 \lceil x\rceil \cdot x^{\lceil x\rceil} - \lfloor x\rfloor\ \mathrm{dx} = \int_0^1 1\cdot x^1 - 0\ \mathrm{dx} = \int_0^1 x\ \mathrm{dx} = \frac{x^2}{2}\bigg|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

$$\int_1^2 \lceil x\rceil \cdot x^{\lceil x\rceil} - \lfloor x\rfloor\ \mathrm{dx} = \int_1^2 2\cdot x^2 - 1\ \mathrm{dx} = \left(\frac{2x^3}{3} - x\right)\bigg|_1^2 = \frac{10}{3} - \left(\frac{-1}{3}\right) = \frac{11}{3}$$

$$\int_2^3 \lceil x\rceil \cdot x^{\lceil x\rceil} - \lfloor x\rfloor\ \mathrm{dx} = \int_1^2 3\cdot x^3 - 2\ \mathrm{dx} = \left(\frac{3x^4}{4} - 2x\right)\bigg|_2^3 = \frac{229}{4} - \left(\frac{32}{4}\right) = \frac{187}{4}$$

Adding these together, we have the following.

$$\frac{1}{2} + \frac{11}{3} + \frac{187}{4} = \frac{6}{12} + \frac{44}{12} + \frac{561}{12} = \boxed{\frac{611}{12}}$$

Problem 16. $\displaystyle\int_0^1 \sin(\sqrt[3]{x})\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{6\sin(1) + 3\cos(1) - 6}$

We now consider the substitution $u = \sqrt[3]{x}$, which implies that $\mathrm{du} = \frac{1}{3}\cdot x^{\frac{-2}{3}}\ \mathrm{dx}$. Thus, we have the following.

$$\int_0^1 \sin(\sqrt[3]{x})\ \mathrm{dx} = \int_0^1 \sin(u)\cdot(3u^2)\ \mathrm{du}$$

For this, we can use tabular integration as follows.

$$\mathscr{D}: 3u^2 \Longrightarrow 6u \Longrightarrow 6 \Longrightarrow 0$$

$$\mathscr{I}: \sin(u) \Longrightarrow -\cos(u) \Longrightarrow -\sin(u) \Longrightarrow \cos(u) \Longrightarrow \sin(u)$$

Thus, our integral evaluates to the following.

$$I = -3u^2\cos(u) + 6u\sin(u) + 6\cos(u)\Big|_0^1$$

$$I = (-3(1)^2\cos(1) + 6(1)\sin(1) + 6\cos(1)) - (-3(0)^2\cos(0) + 6(0)\sin(0) + 6\cos(0))$$

Simplifying, we have the following.

$$I = (-3\cos(1) + 6\sin(1) + 6\cos(1)) - (6\cos(0)) = \boxed{6\sin(1) + 3\cos(1) - 6}$$

Problem 17. $\int_0^{2\pi} \max\{\sin x, \cos x\}\ \mathrm{dx}$

Proposed by Jeck Lim

Solution: $\boxed{2\sqrt{2}}$

We note that $\sin(x) = \cos(x)$ only at the values of $x = \pi/4$ and $x = 5\pi/4$. Thus, we can break up the integration at these points. It is trivial to verify that $\cos(x) > \sin(x)$ on the intervals of $[0, \pi/4]$ and $[5\pi/4, 2\pi]$, with $\sin(x) > \cos(x)$ on the remaining interval of $[\pi/4, 5\pi/4]$. Thus, we have the following resulting integrals.

$$\int_0^{2\pi} \max\{\sin x, \cos x\}\ \mathrm{dx} = \int_0^{\pi/4} \cos(x)\ \mathrm{dx} + \int_{\pi/4}^{5\pi/4} \sin x\ \mathrm{dx} + \int_{5\pi/4}^{2\pi} \cos(x)\ \mathrm{dx}$$

We evaluate each of these as follows.

$$\int_0^{\pi/4} \cos(x)\ \mathrm{dx} + \int_{\pi/4}^{5\pi/4} \sin x\ \mathrm{dx} + \int_{5\pi/4}^{2\pi} \cos(x)\ \mathrm{dx} = (\sin(x))\Big|_0^{\pi/4} + (-\cos(x))\Big|_{\pi/4}^{5\pi/4} + (\sin(x))\Big|_{5\pi/4}^{2\pi}$$

This simplifies as follows.

$$(\sin(x))\Big|_0^{\pi/4} + (-\cos(x))\Big|_{\pi/4}^{5\pi/4} + (\sin(x))\Big|_{5\pi/4}^{2\pi} = \left(\frac{\sqrt{2}}{2} - 0\right) + \left(\frac{\sqrt{2}}{2} - \frac{-\sqrt{2}}{2}\right) + \left(0 - \frac{-\sqrt{2}}{2}\right) = \boxed{2\sqrt{2}}$$

Problem 18. $\int \cos(x)\csc(x)\cot(x)\ \mathrm{dx}$

Proposed by Jeck Lim

Solution: $\boxed{-x - \cot(x) + C}$

We can re-express the integral as follows.

$$\int \cos(x) \cdot \left(\frac{1}{\sin(x)}\right) \cdot \cot(x)\ \mathrm{dx} = \int \cot(x) \cdot \cot(x)\ \mathrm{dx} = \int \cot^2(x)\ \mathrm{dx}$$

Using the Pythagorean Identity, we have that $1+\cot^2(x) = \csc^2(x)$, which we can substitute and simplify as follows.

$$\int \cot^2(x)\ \mathrm{dx} = \int \csc^2(x) - 1\ \mathrm{dx} = \boxed{-x - \cot(x) + C}$$

Problem 19. $\displaystyle\int \frac{-x\cdot e^{\frac{-1}{1-x^2}}}{(1-x^2)^2}\ \mathrm{dx}$

Proposed by Brian Yang

Solution: $\boxed{\dfrac{e^{\frac{-1}{1-x^2}}}{2} + C}$

With the x on the outside of the exponential, we are motivated to consider the u-substitution of $u = 1-x^2$. This implies $\mathrm{du} = -2x\ \mathrm{dx}$, and we have the following equivalent integral.

$$\int \frac{-x\cdot e^{\frac{-1}{1-x^2}}}{(1-x^2)^2}\ \mathrm{dx} = \int \frac{e^{\frac{-1}{u}}\ \mathrm{du}}{2u^2}$$

We can now utilize a secondary u-substitution, namely with $v = -1/u$. This would imply that $\mathrm{dv} = 1/u^2\ \mathrm{du}$, and we have the following equivalent integral.

$$\int \frac{e^{\frac{-1}{u}}\ \mathrm{du}}{2u^2} = \int \frac{e^v\ \mathrm{dv}}{2} = \frac{e^v}{2} + C$$

We now undo the u-substitutions as follows.

$$\frac{e^v}{2} + C = \frac{e^{\frac{-1}{u}}}{2} + C = \boxed{\frac{e^{\frac{-1}{1-x^2}}}{2} + C}$$

Problem 20. $\displaystyle\int \frac{x^{\frac{-1}{2}}}{1+x^{\frac{1}{3}}}\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{6x^{\frac{1}{6}} - 6\arctan\left(x^{\frac{1}{6}}\right) + C}$

We see that there is a $x^{1/2}$ and $x^{1/3}$ present in the integrand. To make the integrand in terms of only whole powers, we consider $u = x^{1/6}$. This would make du as follows.

$$\mathrm{du} = \frac{1}{6}\cdot x^{-5/6}\ \mathrm{dx} = \frac{1}{6}\cdot (x^{1/6})^{-5}\ \mathrm{dx} = \frac{1}{6}\cdot u^{-5}\ \mathrm{dx} \implies 6u^5\ \mathrm{du} = \mathrm{dx}$$

Substituting into the original integral, we have the following.

$$\int \frac{x^{\frac{-1}{2}}}{1+x^{\frac{1}{3}}}\ \mathrm{dx} = \int \frac{\left(x^{\frac{1}{6}}\right)^{-3}}{1+\left(x^{\frac{1}{6}}\right)^2}\ \mathrm{dx} = \int \frac{u^{-3}}{1+u^2}\cdot\left(6u^5\ \mathrm{du}\right) = \int \frac{6u^2}{u^2+1}\ \mathrm{du}$$

Factoring out the 6, we utilize the +0 trick, in which we add 1 and subtract 1 in an attempt to match the denominator and split as follows.

$$6\cdot\left(\int \frac{(u^2+\mathbf{1})-\mathbf{1}}{u^2+1}\ \mathrm{du}\right)=6\cdot\left(\int \frac{u^2+1}{u^2+1}\ \mathrm{du}-\int\frac{1}{u^2+1}\ \mathrm{du}\right)=6\cdot\left(\int 1\ \mathrm{du}-\int\frac{1}{u^2+1}\ \mathrm{du}\right)$$

Both of these integrals are fairly simple to evaluate, but we must remember to substitute back in terms of x to get the final answer.

$$6\cdot\left(\int 1\ \mathrm{du}-\int\frac{1}{u^2+1}\ \mathrm{du}\right)=6\cdot u-6\arctan(u)+C=\boxed{6x^{\frac{1}{6}}-6\arctan\left(x^{\frac{1}{6}}\right)+C}$$

Problem 21. $\displaystyle\int \frac{x^2+\cos^2(x)}{(1+x^2)\sin^2(x)}\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{-\arctan(x)-\cot(x)+C}$

We begin by utilizing the +0 trick, in which we add 1 and subtract 1 in an attempt to match the denominator and split as follows.

$$\int\frac{(x^2+\mathbf{1})+(\cos^2(x)-\mathbf{1})}{(1+x^2)\sin^2(x)}\ \mathrm{dx}=\int\frac{(x^2+1)}{(1+x^2)\sin^2(x)}\ \mathrm{dx}+\int\frac{\cos^2(x)-1}{(1+x^2)\sin^2(x)}\ \mathrm{dx}$$

Simplifying, we have the following.

$$\int\frac{(x^2+1)}{(1+x^2)\sin^2(x)}\ \mathrm{dx}+\int\frac{\cos^2(x)-1}{(1+x^2)\sin^2(x)}\ \mathrm{dx}=\int\frac{1}{\sin^2(x)}\ \mathrm{dx}+\int\frac{-\sin^2(x)}{(1+x^2)\sin^2(x)}\ \mathrm{dx}$$

$$\int\frac{1}{\sin^2(x)}\ \mathrm{dx}+\int\frac{-\sin^2(x)}{(1+x^2)\sin^2(x)}\ \mathrm{dx}=\int\csc^2(x)\ \mathrm{dx}-\int\frac{1}{(1+x^2)}\ \mathrm{dx}$$

Evaluating each of these integrals, we have the final answer as follows.

$$\int\csc^2(x)\ \mathrm{dx}-\int\frac{1}{(1+x^2)}\ \mathrm{dx}=-\cot(x)-\arctan(x)+C=\boxed{-\arctan(x)-\cot(x)+C}$$

Problem 22. $\displaystyle\int\frac{e^{2x}-1}{\sqrt{e^{3x}+e^x}}\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{2\sqrt{e^x+e^{-x}}+C}$ **or** $\boxed{2e^{-x}\sqrt{e^{3x}+e^x}+C}$

We first begin by factoring out e^x from the square root in the denominator as follows.

$$\int\frac{e^{2x}-1}{\sqrt{e^{3x}+e^x}}\ \mathrm{dx}=\int\frac{e^{2x}-1}{e^x\cdot\sqrt{e^x-e^{-x}}}\ \mathrm{dx}=\int\frac{e^x-e^{-x}}{\sqrt{e^x+e^{-x}}}\ \mathrm{dx}$$

We note that the numerator is exactly the derivative of the argument in the square root. Thus, we let $u = e^x + e^{-x}$, which implies $\mathrm{du} = e^x - e^{-x}\ \mathrm{dx}$ and the following equivalent integral.

$$\int \frac{e^x - e^{-x}}{\sqrt{e^x + e^{-x}}}\ \mathrm{dx} = \int \frac{\mathrm{du}}{\sqrt{u}} = 2\sqrt{u} + C = \boxed{2\sqrt{e^x + e^{-x}} + C} = \boxed{2e^{-x}\sqrt{e^{3x} + e^x} + C}$$

Problem 23. $\displaystyle\int \frac{\sin(\ln(x))}{x^3}\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\dfrac{\cos(\ln(x)) + 2\sin(\ln(x))}{-5x^2} + C}$

To remove the $\ln(x)$, we consider the substitution $x = e^u$. This implies that $\mathrm{dx} = e^u\ \mathrm{du}$, and the following equivalent integral.

$$\int \frac{\sin(\ln(x))}{x^3}\ \mathrm{dx} = \int \frac{\sin(\ln(e^u))}{(e^u)^3} \cdot (e^u\ \mathrm{du}) = \int \sin(u) \cdot e^{-2u}\ \mathrm{du}$$

We can use integration by parts on this, with $a = \sin(u)$ and $\mathrm{db} = e^{-2u}\ \mathrm{du}$, which gives the following.

$$\int a\ \mathrm{db} = ab - \int b\ \mathrm{da} = \left(\sin(u) \cdot \frac{e^{-2u}}{-2}\right) - \int \cos(u) \cdot \left(\frac{e^{-2u}}{-2}\right)\ \mathrm{du}$$

Doing integration by parts on this integral again, with $b = \cos(u)$ and $\mathrm{da} = e^{-2u}/(-2)\ \mathrm{du}$, we have the following.

$$\int b\ \mathrm{da} = ab - \int a\ \mathrm{db} = \left(\cos(u) \cdot \frac{e^{-2u}}{4}\right) + \int \sin(u) \cdot \left(\frac{e^{-2u}}{4}\right)\ \mathrm{du}$$

Note that $1/4$ of the original integral has reappeared after this second application of integration by parts. Thus, we have the following, where I is the value of the initial integral.

$$I = \left(\sin(u) \cdot \frac{e^{-2u}}{-2}\right) - \int \cos(u) \cdot \left(\frac{e^{-2u}}{-2}\right)\ \mathrm{du} = \left(\sin(u) \cdot \frac{e^{-2u}}{-2}\right) - \left(\left(\cos(u) \cdot \frac{e^{-2u}}{4}\right) + \frac{I}{4}\right)$$

Combining the like terms of I, we have the following.

$$\frac{5I}{4} = \left(\sin(u) \cdot \frac{e^{-2u}}{-2}\right) - \left(\cos(u) \cdot \frac{e^{-2u}}{4}\right)$$

Substituting u in terms of x, which would be $u = \ln(x)$, we have the following equivalent form.

$$\frac{5I}{4} = \left(\sin(\ln(x)) \cdot \frac{e^{-2\ln(x)}}{-2}\right) - \left(\cos(\ln(x)) \cdot \frac{e^{-2\ln(x)}}{4}\right) = \frac{\sin(\ln(x))}{-2x^2} - \frac{\cos(\ln(x))}{4x^2} = \frac{2\sin(\ln(x))}{-4x^2} - \frac{\cos(\ln(x))}{4x^2}$$

Simplifying, we have the following.

$$I = \frac{4}{5} \cdot \left(\frac{2\sin(\ln(x)) + \cos(\ln(x))}{-4x^2}\right) = \boxed{\frac{\cos(\ln(x)) + 2\sin(\ln(x))}{-5x^2} + C}$$

Problem 24. $\int_0^1 x \cdot \ln^2(x)\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\frac{1}{4}}$

We can consider the substitution $x = e^u$ to remove the logarithm from the integrand as follows, noting that this would also imply that $\mathrm{dx} = e^u\ \mathrm{du}$.

$$\int_0^1 x \cdot \ln^2(x)\ \mathrm{dx} = \int_{-\infty}^0 e^u \cdot (\ln(e^u))^2 \cdot (e^u\ \mathrm{du}) = \int_{-\infty}^0 e^{2u} \cdot u^2\ \mathrm{du}$$

For this, we can use tabular integration as follows.

$$\mathscr{D} : u^2 \Longrightarrow 2u \Longrightarrow 2 \Longrightarrow 0$$

$$\mathscr{I} : e^{2u} \Longrightarrow e^{2u}/2 \Longrightarrow e^{2u}/4 \Longrightarrow e^{2u}/8$$

Thus, our integral evaluates to the following.

$$I = \left(\frac{u^2 e^{2u}}{2} - \frac{(2u) \cdot e^{2u}}{4} + \frac{2 \cdot e^{2u}}{8}\right)\Bigg|_{-\infty}^0 = \left(\frac{u^2 e^{2u}}{2} - \frac{(u) \cdot e^{2u}}{2} + \frac{e^{2u}}{4}\right)\Bigg|_{-\infty}^0$$

$$I = \left(\frac{e^0}{2} - \frac{1 \cdot e^0}{2} + \frac{e^0}{4}\right) - (0 - 0 + 0)$$

Simplifying, we have the following.

$$I = \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{4}\right) - (0) = \boxed{\frac{1}{4}}$$

Problem 25. $\int x^2 \cos\left(\frac{1}{x}\right) + \frac{x}{3} \sin\left(\frac{1}{x}\right)\ \mathrm{dx}$

Proposed by Brian Yang

Solution: $\boxed{\frac{x^3}{3} \cos\left(\frac{1}{x}\right) + C}$

We seek to manipulate the integrand in the form of a product rule, otherwise known as $(fg)' = f'g + g'f$. Since we see that there is a $\cos(1/x)$ and a $\sin(1/x)$ present in the integrand, we claim that $g = \cos(1/x)$. Substituting this into our product rule equation, we have the following.

$$\left(f \cdot \cos\left(\frac{1}{x}\right)\right)' = f' \cdot \cos\left(\frac{1}{x}\right) + f \cdot \left(-\sin\left(\frac{1}{x}\right)\right) \cdot \left(\frac{-1}{x^2}\right) = f' \cdot \cos\left(\frac{1}{x}\right) + \frac{f}{x^2} \cdot \sin\left(\frac{1}{x}\right)$$

Noting the resemblance to the integrand, we let $f = x^3/3$ to match with the second component, which works as follows.

$$\left(\frac{x^3}{3} \cdot \cos\left(\frac{1}{x}\right)\right)' = \left(\frac{x^3}{3}\right)' \cdot \cos\left(\frac{1}{x}\right) + \frac{\frac{x^3}{3}}{x^2} \cdot \sin\left(\frac{1}{x}\right) = x^2 \cdot \cos\left(\frac{1}{x}\right) + \frac{x}{3} \cdot \sin\left(\frac{1}{x}\right)$$

Thus, we can integrate as follows.

$$\int x^2 \cos\left(\frac{1}{x}\right) + \frac{x}{3}\sin\left(\frac{1}{x}\right) = \int \left(\frac{x^3}{3}\cdot\cos\left(\frac{1}{x}\right)\right)' = \boxed{\frac{x^3}{3}\cos\left(\frac{1}{x}\right) + C}$$

Problem 26. $\displaystyle\int_1^e 2\ln(x) + (\ln(x))^2 \text{ dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{e}$

We seek to manipulate the integrand in the form of a product rule, otherwise known as $(fg)' = f'g + g'f$.

$$\int_1^e 2\ln(x) + (\ln(x))^2 \text{ dx} = \int_1^e \left(\frac{2\cdot\ln(x)}{x}\right)\cdot x + (\ln(x))^2 \cdot 1 \text{ dx}$$

We recognize as $f(x) = \ln^2(x)$ and $g(x) = x$, meaning that $f'(x) = \frac{2\ln(x)}{x}$ and $g'(x) = 1$.

$$\int_1^e \left(\frac{2\cdot\ln(x)}{x}\right)\cdot x + (\ln(x))^2\cdot 1 \text{ dx} = \int_1^e f'(x)g(x) + f(x)g'(x) \text{ dx}$$

$$\int_1^e (f(x)g(x))' = f(x)g(x)\Big|_1^e = \ln^2(x)\cdot x\Big|_1^e = \ln^2(e)\cdot e - \ln^2(1)\cdot 1 = \boxed{e}$$

Problem 27. $\displaystyle\int_0^3 \max\left\{\sqrt{1-(x-1)^2}, \sqrt{1-(x-2)^2}\right\} \text{ dx}$

Proposed by Brian Yang

Solution: $\boxed{\frac{\sqrt{3}}{4} + \frac{\pi}{6}}$

The fastest approach to this integral is to solve it geometrically. The two functions represent semicircles (since the square root results in a non-negative value), both with radius 1 and centered respectively at $(1,0)$ and $(2,0)$.

$$\mathscr{C}_1 : y = \sqrt{1-(x-1)^2} \implies (x-1)^2 + y^2 = 1$$

$$\mathscr{C}_2 : y = \sqrt{1-(x-2)^2} \implies (x-2)^2 + y^2 = 1$$

The two semi-circles intersect at $x = 1.5$, and have a shared area. Since we want the maximum of the two functions across the interval, we note that $\mathscr{C}_1$ will be the maximum of the two on the interval of $[1, 1.5]$, and $\mathscr{C}_2$ will be the maximum of the two on the interval of $[1.5, 2]$.

Thus, it suffices to find the area and subtract from π. The area of the overlapping area A is composed of two sectors and a triangle, which we calculate as follows.

$$A = [\text{triangle}] + 2[\text{sector}]$$

The height of the triangle is calculated as $\sqrt{1-(1.5-1)^2} = \sqrt{1-0.5^2} = \sqrt{0.75} = \sqrt{3}/2$. The base of the triangle is 1, since the semi-circles intersect the x-axis at $x=1$ and $x=2$. As for the area of the arcs, they subtend an angle of $\pi/6$ with a radius of 1. Thus, we have the following calculation.

$$A = \frac{\frac{\sqrt{3}}{2}\cdot 1}{2} + 2\cdot\left(\frac{1}{2}\cdot(1)^2\cdot\frac{\pi}{6}\right) = \frac{\sqrt{3}}{4} + 2\cdot\left(\frac{\pi}{12}\right) = \boxed{\frac{\sqrt{3}}{4}+\frac{\pi}{6}}$$

Problem 28. $\int \frac{e^x-1}{e^x+1}\,\mathrm{dx}$

Proposed by Brian Yang

Solution: $\boxed{2\ln(e^{-x}+1)+x+C}$ **or** $\boxed{2\ln(e^x+1)-x+C}$

We begin by utilizing the $+0$ trick, in which we add 2 and subtract -2 in an attempt to match the denominator and split as follows.

$$\int \frac{(e^x-1+\mathbf{2})-\mathbf{2}}{e^x+1}\,\mathrm{dx} = \int \frac{(e^x+1)-2}{e^x+1}\,\mathrm{dx} = \int \frac{e^x+1}{e^x+1}\,\mathrm{dx} - \int \frac{2}{e^x+1}\,\mathrm{dx} = \int 1\,\mathrm{dx} - \int \frac{2}{e^x+1}\,\mathrm{dx}$$

The first integral is trivial, but for the second integral, we consider multiplying both the numerator and denominator by e^{-x} as follows.

$$\int \frac{2}{e^x+1}\cdot\frac{e^{-x}}{e^{-x}}\,\mathrm{dx} = \int \frac{2e^{-x}}{1+e^{-x}}\,\mathrm{dx}$$

We note that the derivative of the denominator is the numerator (ignoring the factor of 2). Thus, we consider $u = 1+e^{-x}$, which implies $\mathrm{du} = -e^{-x}\,\mathrm{dx}$ and the following equivalent integral. Note that we remove the absolute value sign because $e^{-x}+1$ is never negative.

$$\int \frac{2e^{-x}}{1+e^{-x}}\,\mathrm{dx} = \int \frac{-2\,\mathrm{du}}{u} = -2\ln|u| = -2\ln(e^{-x}+1)+C$$

Therefore, the final answer is as follows.

$$\int 1\,\mathrm{dx} - \int \frac{2}{e^x+1}\,\mathrm{dx} = x-(-2\ln(e^{-x}+1))+C = \boxed{2\ln(e^{-x}+1)+x+C}$$

An equivalent form that can be formed is if you factor out e^{-x} as follows.

$$2\ln(e^{-x}(1+e^x))+x+C = 2\cdot[\ln(e^{-x})+\ln(1+e^x)]+x+C = 2\ln(1+e^x)-2x+x+C = \boxed{2\ln(e^x+1)-x+C}$$

Problem 29. $\int_0^1 2^{\lfloor \log_2 x\rfloor}\,\mathrm{dx}$

Proposed by Jeck Lim

Solution: $\boxed{\frac{1}{3}}$

We note for $x\in[1/2,1]$, the following holds true regarding the floor function.

$$\lfloor \log_2 x\rfloor = -1$$

Thus, our integral becomes the following.

$$\int_0^1 2^{\lfloor \log_2 x \rfloor}\, dx = \int_0^{1/2} 2^{\lfloor \log_2 x \rfloor}\, dx + \int_{1/2}^{1} 2^{-1}\, dx = \int_0^{1/2} 2^{\lfloor \log_2 x \rfloor}\, dx + (2^{-1}) \cdot \left(1 - \frac{1}{2}\right) = \int_0^{1/2} 2^{\lfloor \log_2 x \rfloor}\, dx + (2^{-1})^2$$

We can repeat the same procedure for the first integral, noting that for $x \in [1/4, 1/2]$, the following holds true regarding the floor function.

$$\lfloor \log_2 x \rfloor = -2$$

Thus, our integral becomes the following.

$$\int_0^{1/2} 2^{\lfloor \log_2 x \rfloor}\, dx = \int_0^{1/4} 2^{\lfloor \log_2 x \rfloor}\, dx + \int_{1/4}^{1/2} 2^{-2}\, dx = \int_0^{1/4} 2^{\lfloor \log_2 x \rfloor}\, dx + (2^{-1}) \cdot \left(\frac{1}{2} - \frac{1}{4}\right) = \int_0^{1/4} 2^{\lfloor \log_2 x \rfloor}\, dx + (2^{-2})^2$$

We notice the pattern of the integrand in question, namely that we are adding the area of rectangles with height 2^{-i} and length 2^{-i}. Thus, the integral in question is equivalent to the following summation, which is a simple geometric series.

$$\int_0^1 2^{\lfloor \log_2 x \rfloor}\, dx = \sum_{i=1}^{\infty} \left(2^{-i}\right)^2 = \sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^i = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \boxed{\frac{1}{3}}$$

Problem 30. $\displaystyle\int_0^1 \frac{x^2 - 2x}{x^3 + 1}\, dx$

Proposed by Brian Yang

Solution: $\boxed{\ln(2) - \frac{2\pi\sqrt{3}}{9}}$

By partial fraction decomposition, we rewrite the integral as follows.

$$\int_0^1 \frac{x^2 - 2x}{x^3 + 1}\, dx = \int_0^1 \frac{1}{x+1}\, dx - \int_0^1 \frac{1}{x^2 - x + 1}\, dx$$

For the left integral, it evaluates as follows.

$$\int_0^1 \frac{1}{x+1}\, dx = [\ln(x+1)]_0^1 = \ln 2 - \ln 1 = \ln 2$$

The right integral is simplified by completing the square.

$$\int_0^1 \frac{1}{x^2 - x + 1}\, dx = \int_0^1 \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}}\, dx$$

Now we apply a u-substitution of $u = \frac{2x-1}{\sqrt{3}}$, which means $du = \frac{2}{\sqrt{3}}\, dx$:

$$\int_0^1 \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}}\, dx = \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{\sqrt{3}}{2\left(\frac{3u^2}{4} + \frac{3}{4}\right)}\, du = \frac{2\sqrt{3}}{3} \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{1}{u^2 + 1}\, du$$

We can directly compute the integral as follows.

$$\int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{1}{u^2 + 1}\, du = [\arctan(u)]_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}.$$

Thus, we can add all the results of the integrals as follows.

$$\int_0^1 \frac{x^2-2x}{x^3+1}\,\mathrm{dx} = \ln(2) - \frac{2\sqrt{3}}{3}\cdot\left(\frac{\pi}{3}\right) = \boxed{\ln 2 - \frac{2\pi\sqrt{3}}{9}}$$

Problem 31. $\displaystyle\int_0^{\pi/4} \frac{\tan(x)+2\sec^2(x)+2\tan(x)\sec^2(x)}{\tan(x)+\sec^2(x)}\,\mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\dfrac{\pi}{4}+\ln(3)}$

We hope to manipulate the numerator to resemble the denominator. Splitting the $2\sec^2(x) = \sec^2(x)+\sec^2(x)$, we have the following.

$$\int_0^{\pi/4} \frac{(\tan(x)+\sec^2(x))+(\sec^2(x)+2\tan(x)\sec^2(x))}{\tan(x)+\sec^2(x)}\,\mathrm{dx} = \int_0^{\pi/4}\frac{\tan(x)+\sec^2(x)}{\tan(x)+\sec^2(x)}\,\mathrm{dx} + \int_0^{\pi/4}\frac{\sec^2(x)+2\tan(x)\sec^2(x)}{\tan(x)+\sec^2(x)}\,\mathrm{dx}$$

Simplifying, we have the following.

$$\int_0^{\pi/4}\frac{\tan(x)+\sec^2(x)}{\tan(x)+\sec^2(x)}\,\mathrm{dx} + \int_0^{\pi/4}\frac{\sec^2(x)+2\tan(x)\sec^2(x)}{\tan(x)+\sec^2(x)}\,\mathrm{dx} = \int_0^{\pi/4} 1\,\mathrm{dx} + \int_0^{\pi/4}\frac{\sec^2(x)+2\tan(x)\sec^2(x)}{\tan(x)+\sec^2(x)}\,\mathrm{dx}$$

The first integral simplifies as follows.

$$\int_0^{\pi/4} 1\,\mathrm{dx} = x\Big|_0^{\pi/4} = \frac{\pi}{4}$$

For the second integral, we note that the numerator is indeed the derivative of the denominator, as we verify below.

$$\frac{d}{dx}(\tan(x)+\sec^2(x)) = \sec^2(x)+2\sec(x)\cdot(\sec(x)\tan(x)) = \sec^2(x)+2\sec^2(x)\tan(x)$$

Thus, we consider $u=\tan(x)+\sec^2(x)$, which transforms the integral as follows upon changing the bounds.

$$\int_0^{\pi/4}\frac{\sec^2(x)+2\tan(x)\sec^2(x)}{\tan(x)+\sec^2(x)}\,\mathrm{dx} = \int_1^3 \frac{\mathrm{du}}{u} = \ln|u|\Big|_1^3 = \ln(3)-\ln(1) = \ln(3)$$

Thus, the integral becomes the following by adding the results we obtained above.

$$\int_0^{\pi/4} \frac{\tan(x)+2\sec^2(x)+2\tan(x)\sec^2(x)}{\tan(x)+\sec^2(x)}\,\mathrm{dx} = \boxed{\frac{\pi}{4}+\ln(3)}$$

Problem 32. $\displaystyle\int_0^1 (x^6+x^4+x^2)\cdot\sqrt{2x^4+3x^2+6}\,\mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\dfrac{11\sqrt{11}}{18}}$

The first instinct for this integral is to use a u-substitution. However, we see that the degree will not match if we let u be the argument of the square root. Thus, we consider factoring out x from the expression $(x^6+x^4+x^2)$ and bringing it inside the square root as follows.

$$\int_0^1 (x^5+x^3+x^1)\cdot(x)\cdot\sqrt{2x^4+3x^2+6}\,\mathrm{dx} = \int_0^1 (x^5+x^3+x^1)\cdot\sqrt{x^2\cdot(2x^4+3x^2+6)}\,\mathrm{dx}$$

Simplifying, we have the following.

$$\int_0^1 (x^5+x^3+x^1)\cdot\sqrt{x^2\cdot(2x^4+3x^2+6)}\,\mathrm{dx} = \int_0^1 (x^5+x^3+x^1)\cdot\sqrt{2x^6+3x^4+6x^2}\,\mathrm{dx}$$

If we now consider $u = 2x^6+3x^4+6x^2$, we then get that $\mathrm{du} = 12x^5+12x^3+12x\,\mathrm{dx} = 12(x^5+x^3+x)\,\mathrm{dx}$. This is exactly a multiple of the expression outside the square root, so we perform the u-substitution and change the bounds as follows.

$$\int_0^1 (x^6+x^4+x^2)\cdot\sqrt{2x^4+3x^2+6}\,\mathrm{dx} = \int_0^{11}\frac{\sqrt{u}}{12}\,\mathrm{du} = \frac{1}{12}\cdot\frac{2}{3}u^{3/2}\Big|_0^{11} = \frac{11^{3/2}}{18}-\frac{0^{3/2}}{18} = \boxed{\frac{11\sqrt{11}}{18}}$$

Problem 33. $\int_0^3 (x^2+1)\,\mathrm{d}\lfloor x\rfloor$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{17}$

We begin by using integration by parts in order to remove the modified dx component, with $u = x^2+1$ and $\mathrm{d}v = \mathrm{d}\lfloor x\rfloor$. This implies that $\mathrm{du} = 2x\,\mathrm{dx}$ and $v = \lfloor x\rfloor$. Thus, we get the following equivalent expression as follows.

$$\int_0^3 (x^2+1)\,\mathrm{d}\lfloor x\rfloor = \int u\,\mathrm{dv} = uv - \int v\,\mathrm{du} = (x^2+1)\lfloor x\rfloor\Big|_0^3 - \int_0^3 \lfloor x\rfloor(2x)\,\mathrm{dx}$$

Evaluating the first portion, we get the following.

$$(x^2+1)\lfloor x\rfloor\Big|_0^3 = (3^2+1)\lfloor 3\rfloor - (0^2+1)\lfloor 0\rfloor = 10\cdot 3 - 1\cdot 0 = 30$$

As for the resulting integral, $\lfloor x\rfloor$ will be an integer across the different intervals. Thus, we break up the integral onto the respective intervals as follows.

$$\int_0^3 \lfloor x\rfloor(2x)\,\mathrm{dx} = \int_0^1 \lfloor x\rfloor(2x)\,\mathrm{dx} + \int_1^2 \lfloor x\rfloor(2x)\,\mathrm{dx} + \int_2^3 \lfloor x\rfloor(2x)\,\mathrm{dx}$$

We note that $\lfloor x\rfloor = \{0,1,2\}$ on the given intervals, respectively. Substituting this in, we have the following.

$$\int_0^1 \lfloor x\rfloor(2x)\,\mathrm{dx} = \int_0^1 0\cdot(2x)\,\mathrm{dx} = 0$$

$$\int_1^2 \lfloor x\rfloor(2x)\,\mathrm{dx} = \int_1^2 2x\,\mathrm{dx} = x^2\Big|_1^2 = 2^2-1^2 = 3$$

$$\int_2^3 \lfloor x \rfloor (2x)\ \mathrm{dx} = \int_2^3 2\cdot 2x = 2x^2\Big|_2^3 = 2(3)^2 - 2(2)^2 = 18 - 8 = 10$$

Thus, the final integral becomes the following.

$$30 - \int_0^3 \lfloor x \rfloor (2x)\ \mathrm{dx} = 30 - (0+3+10) = 30 - 13 = \boxed{17}$$

Problem 34. $\int_0^\pi \frac{1-\sin x}{1+\sin x}\ \mathrm{dx}$

Proposed by Jeck Lim

Solution: $\boxed{4-\pi}$

We immediately consider multiplying by the conjugate of $1-\sin(x)$, which is $1+\sin(x)$. This will force the use of the Pythagorean identity in the denominator as follows.

$$\int_0^\pi \frac{1-\sin(x)}{1+\sin(x)}\cdot\frac{1-\sin(x)}{1-\sin(x)}\ \mathrm{dx} = \int_0^\pi \frac{(1-\sin(x))^2}{1-\sin^2(x)}\ \mathrm{dx} = \int_0^\pi \frac{1-2\sin(x)+\sin^2(x)}{\cos^2(x)}\ \mathrm{dx}$$

Dividing each of the components by $\cos^2(x)$, we get the following.

$$\int_0^\pi \frac{1-2\sin(x)+\sin^2(x)}{\cos^2(x)}\ \mathrm{dx} = \int_0^\pi \sec^2(x) - 2\tan(x)\sec(x) + \tan^2(x)\ \mathrm{dx}$$

Re-expressing $\tan^2(x) = \sec^2(x) - 1$ to bring the terms of the integrand in terms of $\sec^2(x)$, we have the following.

$$\int_0^\pi \sec^2(x) - 2\tan(x)\sec(x) + (\sec^2(x)-1)\ \mathrm{dx} = \int_0^\pi 2\sec^2(x) - 2\tan(x)\sec(x) - 1\ \mathrm{dx}$$

These are simple trigonometric anti-derivatives, which are shown below.

$$\int_0^\pi 2\sec^2(x) - 2\tan(x)\sec(x) - 1\ \mathrm{dx} = (2\tan(x) - 2\sec(x) - x)\Big|_0^\pi$$

Simplifying, we have the following.

$$(2\tan(x) - 2\sec(x) - x)\Big|_0^\pi = (2\tan(\pi) - 2\sec(\pi) - \pi) - (2\tan(0) - 2\sec(0) - 0) = (0-(-2)-\pi) - (0-2-0) = \boxed{4-\pi}$$

Problem 35. $\int \frac{e^{3x}(6x-5)}{(2x-1)^2}\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\frac{e^{3x}}{2x-1} + C}$

We seek to manipulate the integrand in the form of a product rule, otherwise known as $(fg)' = f'g + g'f$.

Since we see that there is a e^{3x} present in the integrand, we claim that $g = e^{3x}$. This is because upon taking derivative, e^{3x} will still be present Substituting this into our product rule equation, we have the following.

$$\left(f \cdot e^{3x}\right)' = f' \cdot e^{3x} + f \cdot 3e^{3x} = e^{3x} \cdot (f' + 3f)$$

Since this must match the integrand, we have the following.

$$\frac{e^{3x}(6x-5)}{(2x-1)^2} = e^{3x} \cdot (f' + 3f) \implies \frac{6x-5}{(2x-1)^2} = f' + 3f$$

We begin by utilizing the $+0$ trick, in which we add 2 and subtract -2 in an attempt to match the denominator and split as follows.

$$\frac{6x-5+\mathbf{2}-\mathbf{2}}{(2x-1)^2} = \frac{(6x-3)-2}{(2x-1)^2} = \frac{3(2x-1)-2}{(2x-1)^2} = \frac{3(2x-1)}{(2x-1)^2} - \frac{2}{(2x-1)^2} = \frac{3}{2x-1} - \frac{2}{(2x-1)^2} = f' + 3f$$

From this, it is clear that we select $f = 1/(2x-1)$. We now verify that the derivative is as follows.

$$f' = ((2x-1)^{-1})' = -((2x-1)^{-2}) \cdot 2 = \frac{-2}{(2x-1)^2}$$

Thus, we now have the desired product rule, which we can integrate as follows.

$$\int \frac{e^{3x}(6x-5)}{(2x-1)^2}\,\mathrm{dx} = \int \left(\frac{1}{2x-1} \cdot e^{3x}\right)' \mathrm{dx} = \boxed{\frac{e^{3x}}{2x-1} + C}$$

Problem 36. $\displaystyle\int_1^2 \frac{9x+4}{x^5+3x^2+x}\,\mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\ln\left(\frac{80}{23}\right)}$

We begin by doing partial fraction decomposition on the integrand as follows.

$$\int_1^2 \frac{9x+4}{x \cdot (x^4+3x+1)}\,\mathrm{dx} = \int_1^2 \frac{A}{x} + \frac{Bx^3+Cx^2+Dx+E}{x^4+3x+1}\,\mathrm{dx}$$

Cross multiplying, we have the following.

$$9x+4 = A(x^4+3x+1) + x \cdot (Bx^3+Cx^2+Dx+E) = (A+B)x^4 + Cx^3 + Dx^2 + (3A+E)x + A$$

Matching the polynomial components, it is clear that $A = 4$, $B = -4$, $C = 0$, $D = 0$, and $E = -3$. Thus, our integral becomes the following.

$$\int_1^2 \frac{A}{x} + \frac{Bx^3+Cx^2+Dx+E}{x^4+3x^2+1}\,\mathrm{dx} = \int_1^2 \frac{4}{x} + \frac{-4x^3-3}{x^4+3x+1}\,\mathrm{dx} = \int_1^2 \frac{4}{x}\,\mathrm{dx} - \int_1^2 \frac{4x^3+3}{x^4+3x+1}\,\mathrm{dx}$$

The first integral is trivial to evaluate as follows.

$$\int_1^2 \frac{4}{x}\,\mathrm{dx} = 4\ln|x|\Big|_1^2 = 4\ln(2) - 4\ln(1) = \ln(2^4) - 0 = \ln(16)$$

For the second integral, we realize that the numerator is indeed the derivative of the denominator. Thus, we consider the u-substitution of $u = x^4 + 3x + 1$, which implies that $\mathrm{du} = 4x^3 + 3$. Thus, we have the following equivalent integral.

$$\int_1^2 \frac{4x^3+3}{x^4+3x+1}\,\mathrm{dx} = \int_{(1)^4+3(1)+1}^{(2)^4+3(2)+1} \frac{\mathrm{du}}{u} = \int_5^{23} \frac{\mathrm{du}}{u} = \ln|u|\Big|_5^{23} = \ln(23) - \ln(5) = \ln\left(\frac{23}{5}\right)$$

Thus, the final result of the integral becomes the following.

$$\int_1^2 \frac{4}{x}\,\mathrm{dx} - \int_1^2 \frac{4x^3+3}{x^4+3x+1}\,\mathrm{dx} = \ln(16) - \ln\left(\frac{23}{5}\right) = \ln\left(\frac{16}{\frac{23}{5}}\right) = \ln\left(\frac{16\cdot 5}{23}\right) = \boxed{\ln\left(\frac{80}{23}\right)}$$

Problem 37. $\int \ln(x^2+1)\,\mathrm{dx}$

Proposed by Jeck Lim

Solution: $\boxed{x\cdot\ln(x^2+1) - 2x + 2\arctan(x) + C}$

We begin by utilizing integration by parts, with $u = \ln(x^2+1)$ and $\mathrm{dv} = \mathrm{dx}$. Thus, we have the following equivalent expression.

$$\int u\,\mathrm{dv} = uv - \int v\,\mathrm{du} = \ln(x^2+1)\cdot x - \int \frac{x\cdot 2x}{x^2+1}\,\mathrm{dx}$$

For the integral, we factor out the 2, we utilize the +0 trick, in which we add 1 and subtract 1 in an attempt to match the denominator and split as follows.

$$2\cdot\left(\int \frac{(x^2+\mathbf{1})-\mathbf{1}}{x^2+1}\,\mathrm{dx}\right) = 2\cdot\left(\int \frac{x^2+1}{x^2+1}\,\mathrm{dx} - \int \frac{1}{x^2+1}\,\mathrm{dx}\right) = 2\cdot\left(\int 1\,\mathrm{dx} - \int \frac{1}{x^2+1}\,\mathrm{dx}\right)$$

Both of these integrals are fairly simple to evaluate, but we must remember to substitute back in terms of x to get the final answer.

$$2\cdot\left(\int 1\,\mathrm{dx} - \int \frac{1}{x^2+1}\,\mathrm{dx}\right) = 2x - 2\arctan(x) + C$$

Thus, the final integral becomes the following.

$$\ln(x^2+1)\cdot x - \int \frac{x\cdot 2x}{x^2+1}\,\mathrm{dx} = x\ln(x^2+1) - (2\cdot x - 2\arctan(x)) + C = \boxed{x\cdot\ln(x^2+1) - 2x + 2\arctan(x) + C}$$

Problem 38. $\int \frac{(x^2+1)(x^2+4)}{(x^2+2)(x^2+3)}\,\mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{x - \sqrt{2}\cdot\arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{2}{\sqrt{3}}\cdot\arctan\left(\frac{x}{\sqrt{3}}\right)}$

Expanding the numerator and denominator, we have the following.

$$\int \frac{(x^2+1)(x^2+4)}{(x^2+2)(x^2+3)} = \int \frac{x^4+5x^2+4}{x^4+5x^2+6}$$

We begin by utilizing the $+0$ trick, in which we add 2 and subtract -2 in an attempt to match the denominator and split as follows.

$$\int \frac{(x^4+5x^2+4+\mathbf{2})-\mathbf{2}}{x^4+5x^2+6}\,\mathrm{dx} = \int \frac{(x^4+5x^2+6)-2}{x^4+5x^2+6}\,\mathrm{dx} = \int \frac{x^4+5x^2+6}{x^4+5x^2+6}\,\mathrm{dx} - \int \frac{2}{x^4+5x^2+6}\,\mathrm{dx}$$

Simplifying, we have the following.

$$\int \frac{x^4+5x^2+6}{x^4+5x^2+6}\,\mathrm{dx} - \int \frac{2}{x^4+5x^2+6}\,\mathrm{dx} = \int 1\,\mathrm{dx} - \int \frac{2}{x^4+5x^2+6}\,\mathrm{dx} = x + \int \frac{-2}{x^4+5x^2+6}\,\mathrm{dx}$$

We begin by doing partial fraction decomposition on the integrand as follows.

$$\int \frac{-2}{x^4+5x^2+6}\,\mathrm{dx} = \int \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+3}\,\mathrm{dx}$$

Cross multiplying, we have the following.

$$-2 = (Ax+B)(x^2+3)+(Cx+D)(x^2+2) = (A+C)x^3+(B+D)x^2+(3A+2C)x+(3B+2D)$$

Since $A+C=0$ and $3A+2C=0$, it must be the case that $A=C=0$. Additionally, we are told that $B+D=0$ and $3B+2D=2$, which implies that $D=2$ and $B=-2$. Thus, we have the following equivalent integral.

$$\int \frac{-2}{x^4+5x^2+6}\,\mathrm{dx} = \int \frac{-2}{x^2+2} + \frac{2}{x^2+3}\,\mathrm{dx} = \int \frac{-2}{x^2+2}\,\mathrm{dx} + \int \frac{2}{x^2+3}\,\mathrm{dx}$$

We can calculate each of these integrals as follows, using their resemblance to the derivative of $\arctan(x)$.

$$\int \frac{-2}{x^2+2}\cdot\frac{1/2}{1/2}\,\mathrm{dx} = \int \frac{-1}{\frac{x^2}{2}+1}\,\mathrm{dx} = \int \frac{-1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\,\mathrm{dx} = \sqrt{2}\cdot\arctan\left(\frac{x}{\sqrt{2}}\right)+C$$

$$\int \frac{2}{x^2+3}\cdot\frac{1/3}{1/3}\,\mathrm{dx} = \int \frac{2/3}{\frac{x^2}{3}+1}\,\mathrm{dx} = \int \frac{2/3}{\left(\frac{x}{\sqrt{3}}\right)^2+1}\,\mathrm{dx} = \frac{2}{\sqrt{3}}\cdot\arctan\left(\frac{x}{\sqrt{3}}\right)+C$$

Thus, the integral becomes the following.

$$x + \int \frac{-2}{x^4+5x^2+6}\,\mathrm{dx} = \boxed{x-\sqrt{2}\cdot\arctan\left(\frac{x}{\sqrt{2}}\right)+\frac{2}{\sqrt{3}}\cdot\arctan\left(\frac{x}{\sqrt{3}}\right)}$$

Problem 39. $\displaystyle\int_0^1 \lfloor \log_{2023} x \rfloor\,\mathrm{dx}$

Proposed by Brian Yang

Solution: $\boxed{-\frac{2023}{2022}}$

We note for $x \in [1/2023, 1]$, the following holds true regarding the floor function.

$$\lfloor \log_{2023} x \rfloor = -1$$

Thus, our integral becomes the following.

$$\int_0^1 \lfloor \log_{2023} x \rfloor \, \mathrm{dx} = \int_0^{1/2023} \lfloor \log_{2023} x \rfloor \, \mathrm{dx} + \int_{1/2023}^1 -1 \, \mathrm{dx} = \int_0^{1/2023} \lfloor \log_{2023} x \rfloor \, \mathrm{dx} + (-1) \cdot \left(1 - \frac{1}{2023}\right)$$

$$\int_0^1 \lfloor \log_{2023} x \rfloor \, \mathrm{dx} = \int_0^{1/2023} \lfloor \log_{2023} x \rfloor \, \mathrm{dx} + (-1) \cdot \left(1 - \frac{1}{2023}\right) = \int_0^{1/2023} \lfloor \log_{2023} x \rfloor \, \mathrm{dx} + \frac{-2022}{2023}$$

We can repeat the same procedure for the first integral, noting that for $x \in [1/2023^2, 1/2023]$, the following holds true regarding the floor function.

$$\lfloor \log_{2023} x \rfloor = -2$$

Thus, our integral becomes the following.

$$\int_0^{1/2023} \lfloor \log_{2023} x \rfloor \, \mathrm{dx} = \int_0^{1/2023^2} \lfloor \log_{2023} x \rfloor \, \mathrm{dx} + \int_{1/2023^2}^{1/2023} -2 \, \mathrm{dx} = \int_0^{1/2023^2} \lfloor \log_{2023} x \rfloor \, \mathrm{dx} + (-2) \cdot \left(\frac{1}{2023} - \frac{1}{2023^2}\right)$$

$$\int_0^{1/2023} \lfloor \log_{2023} x \rfloor \, \mathrm{dx} = \int_0^{1/2023^2} \lfloor \log_{2023} x \rfloor \, \mathrm{dx} + (-2) \cdot \left(\frac{1}{2023} - \frac{1}{2023^2}\right) = \int_0^{1/2023^2} \lfloor \log_{2023} x \rfloor \, \mathrm{dx} + \frac{-2 \cdot 2022}{2023^2}$$

We notice the pattern of the integrand in question, namely that we are adding the area of rectangles with height $-i$ and length $(2022)/(2023)^i$. Thus, the integral in question is equivalent to the following summation, which is a simple geometric series.

$$I = \int_0^1 \lfloor \log_{2023} x \rfloor \, \mathrm{dx} = \sum_{i=1}^{\infty} (-i) \cdot \frac{2022}{2023^i} = (-2022) \cdot \sum_{i=1}^{\infty} \frac{i}{2023^i} = (-2022) \cdot S$$

If we let S be the summation, we consider multiplying by $1/2023$ and subtracting as follows.

$$S = \frac{1}{2023} + \frac{2}{2023^2} + \frac{3}{2023^3} + \cdots$$

$$\frac{S}{2023} = \frac{1}{2023^2} + \frac{2}{2023^3} + \frac{3}{2023^4} + \cdots$$

$$\frac{2022S}{2023} = \frac{1}{2023} + \left(\frac{2}{2023^2} - \frac{1}{2023^2}\right) + \left(\frac{3}{2023^3} - \frac{2}{2023^3}\right) = \frac{1}{2023} + \frac{1}{2023^2} + \frac{1}{2023^3} + \cdots$$

$$\frac{2022S}{2023} = \frac{\frac{1}{2023}}{1 - \frac{1}{2023}} = \frac{\frac{1}{2023}}{\frac{2022}{2023}} = \frac{1}{2022} \implies S = \frac{2023}{2022^2}$$

Substituting this for our summation, we have the following.

$$I = (-2022) \cdot S = -2022 \cdot \left(\frac{2023}{2022^2}\right) = \boxed{\frac{-2023}{2022}}$$

Problem 40. $\displaystyle\int \frac{x-1}{\sqrt{2x^2-3}} \, \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\dfrac{\sqrt{2x^2-3}}{2}-\dfrac{\ln\left|\sqrt{\frac{2x^2}{3}-1}+\frac{\sqrt{6}\cdot x}{\sqrt{3}}\right|}{\sqrt{2}}+C}$ **or** $\boxed{\dfrac{\sqrt{2x^2-3}}{2}-\dfrac{\ln|\sqrt{2x^2-3}+\sqrt{2}x|}{\sqrt{2}}+C}$

We begin by splitting the integral as follows.

$$\int \frac{x-1}{\sqrt{2x^2-3}}\,\mathrm{dx} = \int \frac{x}{\sqrt{2x^2-3}}\,\mathrm{dx} - \int \frac{1}{\sqrt{2x^2-3}}\,\mathrm{dx}$$

The first integral can be solved using a u-substitution of $u = 2x^2-3$, which implies $\mathrm{du} = 4x\ \mathrm{dx}$ as follows.

$$\int \frac{x}{\sqrt{2x^2-3}}\,\mathrm{dx} = \int \frac{\mathrm{du}}{4\sqrt{u}} = \frac{\sqrt{u}}{2}+C = \frac{\sqrt{2x^2-3}}{2}+C$$

For the second integral, we consider the trigonometric substitution of $x = \sqrt{3}\sec(\theta)/\sqrt{2}$, which implies that $\mathrm{dx} = (\sqrt{3}\sec(\theta)\tan(\theta))/\sqrt{2}\ \mathrm{d}\theta$. Therefore, we have the following equivalent integral.

$$\int \frac{1}{\sqrt{2x^2-3}}\,\mathrm{dx} = \int \frac{1}{\sqrt{2\left(\frac{\sqrt{3}\sec(\theta)}{\sqrt{2}}\right)^2-3}}\cdot\frac{\sqrt{3}\sec(\theta)\tan(\theta)}{\sqrt{2}}\,\mathrm{d}\theta = \int \frac{\sqrt{3}\sec(\theta)\tan(\theta)}{\sqrt{2}\cdot\sqrt{3\sec^2(\theta)-3}}\,\mathrm{d}\theta$$

Simplifying using the Pythagorean identity, we have the following.

$$\int \frac{\sqrt{3}\sec(\theta)\tan(\theta)}{\sqrt{2}\cdot\sqrt{3\sec^2(\theta)-3}}\,\mathrm{d}\theta = \int \frac{\sqrt{3}\sec(\theta)\tan(\theta)}{\sqrt{2}\cdot\sqrt{3}\tan(\theta)} = \int \frac{1}{\sqrt{2}}\cdot\sec(\theta)\,\mathrm{d}\theta$$

Thus, the integral becomes as follows.

$$\int \frac{1}{\sqrt{2}}\cdot\sec(\theta)\,\mathrm{d}\theta = \frac{\ln|\sec(\theta)+\tan(\theta)|}{\sqrt{2}}+C$$

We need to re-formulate the answer in terms of x, so we can relate the angles as follows.

$$\tan^2(\theta)+1 = \sec^2(\theta)$$

$$\tan^2(\theta)+1 = \left(\frac{\sqrt{6}}{3}\cdot x\right)^2 \implies \tan(\theta) = \sqrt{\frac{2x^2}{3}-1}$$

Thus, the second integral becomes the following.

$$\frac{\ln|\sec(\theta)+\tan(\theta)|}{\sqrt{2}}+C = \frac{\ln\left|\sqrt{\frac{2x^2}{3}-1}+\frac{\sqrt{6}\cdot x}{\sqrt{3}}\right|}{\sqrt{2}}+C$$

Thus, the final answer is as follows.

$$\int \frac{x-1}{\sqrt{2x^2-3}}\,\mathrm{dx} = \int \frac{x}{\sqrt{2x^2-3}}\,\mathrm{dx} - \int \frac{1}{\sqrt{2x^2-3}}\,\mathrm{dx} = \boxed{\frac{\sqrt{2x^2-3}}{2}-\frac{\ln\left|\sqrt{\frac{2x^2}{3}-1}+\frac{\sqrt{6}\cdot x}{\sqrt{3}}\right|}{\sqrt{2}}+C}$$

This can be simplified by factoring out the $\sqrt{3}$ from the $\ln()$ inside the second portion of the anti-derivative as follows.

$$\frac{\sqrt{2x^2-3}}{2}-\frac{\ln\left|\sqrt{3}\cdot\sqrt{2x^2-3}+\sqrt{3}\cdot(\sqrt{2}\cdot x)\right|}{\sqrt{2}}+C=\frac{\sqrt{2x^2-3}}{2}-\frac{\ln\left|\sqrt{2x^2-3}+(\sqrt{2}\cdot x)\right|}{\sqrt{2}}-\ln(\sqrt{3})+C$$

Since $\ln(\sqrt{3})$ is a constant, it can be absorbed into the constant of integration, which results in the equivalent form.

$$\frac{\sqrt{2x^2-3}}{2}-\frac{\ln\left|\sqrt{2x^2-3}+(\sqrt{2}\cdot x)\right|}{\sqrt{2}}-\ln(\sqrt{3})+C=\boxed{\frac{\sqrt{2x^2-3}}{2}-\frac{\ln|\sqrt{2x^2-3}+\sqrt{2}x|}{\sqrt{2}}+C}$$