Caltech Harvey Mudd Mathematics Competition

1. Let a_k be the number of ordered 10-tuples $(x_1, x_2, \dots x_{10})$ of nonnegative integers such that

$$x_1^2 + x_2^2 + \cdots + x_{10}^2 = k.$$

Let $b_k = 0$ if a_k is even and $b_k = 1$ if a_k is odd. Find $\sum_{i=1}^{2012} b_{4i}$.

Solution: The answer is 22. We can pair each 10-tuple with its reverse without changing the parity of the number of solutions. Only those 10-tuples which are palindromes are unpaired, so $x_i = x_{11-i}$ and so we can consider the following equation without changing the b_k sequence:

$$2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 + 2x_5^2 = k.$$

The same trick works again, reducing the problem further to $4x_1^2 + 4x_2^2 + 2x_3^2 = k$. Finally, we can do another pairing by swapping x_1 and x_2, so it suffices to consider those solutions with $x_1 = x_2$ and the equation becomes $8x_1^2 + 2x_3^2 = k$. Notice that we are only considering k that are multiples of 4. Let $k = 4m$. Then we have $(2x_1)^2 + x_3^2 = 2m$. Since $2x_1$ and $2m$ are even, x_3 is even, so let $x_3 = 2y$ and $2x_1 = 2x$. Then the equation reduces to $x^2 + y^2 = \frac{m}{2}$. Finally, since x, y are arbitrary, we can do the same swapping trick to reduce the problem to $x = y$, or $x^2 = \frac{m}{4}$.

For nonnegative integer x, this has an odd number of solutions if and only if $\frac{m}{4}$ is a perfect square. Therefore, the only terms in the sum $\sum_{m=1}^{2012} b_{4m}$ that are 1 are those where $4m$ is a multiple of 16 and a perfect square. That is, m must be an even square. Note that $44^2 < 2012 < 45^2$, so $2^2, 4^2, \dots 44^2$ are the values of m for which $b_{4m} = 1$. There are 22 such values, so the sum evaluates to 22.

2. A convex octahedron in Cartesian space contains the origin in its interior. Two of its vertices are on the x-axis, two are on the y-axis, and two are on the z-axis. One triangular face F has side lengths $\sqrt{17}, \sqrt{37}, \sqrt{52}$. A second triangular face F' has side lengths $\sqrt{13}, \sqrt{29}, \sqrt{34}$. What is the minimum possible volume of the octahedron?

Solution: The answer is $\frac{77}{2}$. Each face of the octahedron contains a vertex on the x-axis, a vertex on the y-axis, and a vertex on the z-axis. Let the distances of these three vertices from the origin be a, b, c. Then the side lengths of the face are $\sqrt{a^2 + b^2}, \sqrt{a^2 + c^2}, \sqrt{b^2 + c^2}$. For face F, let $a_1 < b_1 < c_1$ be the three vertex distances. By squaring each side and summing them together we get $a_1^2 + b_1^2 + c_1^2 = 53$. Then

$$c_1 = \sqrt{53 - (a_1^2 + b_1^2)} = \sqrt{53 - 17} = 6,$$

$$b_1 = \sqrt{53 - (a_1^2 + c_1^2)} = \sqrt{53 - 37} = 4,$$

$$a_1 = \sqrt{53 - (b_1^2 + c_1^2)} = \sqrt{53 - 52} = 1.$$

Through similar methods, the distances of the three vertices of F' are $2, 3, 5$, which implies that F' shares no vertices or edges in common with F and tells us all six distances between each vertex and the origin. It remains to find the configuration that minimizes the volume.

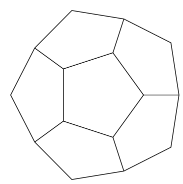
Let d_x, d_y, d_z be the distance between the two x-axis vertices, two y-axis vertices, and two z-axis vertices respectively. By breaking the octahedron into two pyramids cut by one of the coordinate planes, one can quickly find that the volume is $\frac{d_x d_y d_z}{6}$. So we have to minimize the product d_x, d_y, d_z. We also know that

$$d_x = d_{x,1} + d_{x,2}, \qquad d_y = d_{y,1} + d_{y,2}, \qquad d_z = d_{z,1} + d_{z,2},$$

where $\{d_{x,1}, d_{y,1}, d_{z,1}\} = \{1, 4, 6\}$ and $\{d_{x,2}, d_{y,2}, d_{z,2}\} = \{2, 3, 5\}$. Using the rearrangement inequality or other simpler methods of reasoning, we can determine that this product is minimal when we pair the kth highest term in one set with the kth highest term in the other. So we set the three distances d_x, d_y, d_z equal to $6 + 5, 4 + 3, 1 + 2$. This gives us the volume $\frac{(6+5)(4+3)(1+2)}{6} = \frac{77}{2}$.

3. Three different faces of a regular dodecahedron are selected at random and painted. What is the probability that there is at least one pair of painted faces that share an edge?

   Solution: The answer is $\frac{10}{11}$. We do this by complementary counting, finding the probability that no two painted faces are adjacent. Regardless of what the first face chosen is, there will be six faces left that can be chosen which are in the configuration below.



   We need the number of ways to select two of these faces such that they are not adjacent. Clearly if the central face is selected there is no way to select a third face, so both faces must come from the 5-cycle on the outside. Whichever face of the 5-cycle we choose second, the third has two options. So accounting for all three faces, there are $12 \cdot 5 \cdot 2$ options. The total number of options is $12 \cdot 11 \cdot 10$, so the answer is $1 - \frac{12 \cdot 5 \cdot 2}{12 \cdot 11 \cdot 10} = \frac{10}{11}$.

4. The expression below has six empty boxes. Each box is to be filled in with a number from 1 to 6, where all six numbers are used exactly once, and then the expression is evaluated. What is the maximum possible final result that can be achieved?

$$\frac{\frac{\square}{\square} + \frac{\square}{\square}}{\frac{\square}{\square}}$$

   Solution: The answer is 23. It is equivalent to maximize the expression $\frac{\square}{\square}(\frac{\square}{\square} + \frac{\square}{\square})$. Call this $\frac{a}{b}(\frac{c}{d} + \frac{e}{f})$. Notice that if $b \neq 1$, then we can increase the expression by swapping b with whichever variable is 1. Similarly, if $a \neq 6$, we increase the expression by exchanging a with the variable equal to 6. To maximize $\frac{c}{d} + \frac{e}{f}$, we first notice that d, f are $2, 3$ in some order, since otherwise there is a way to do a swap that decreases a denominator and increases a numerator. WLOG $d = 2, f = 3$, so we have $\frac{c}{2} + \frac{e}{3}$. Since $\frac{1}{3} < \frac{1}{2}$, we set c to be greater, so $c = 5, e = 4$. This gets us

$$\frac{6}{1}\left(\frac{5}{2} + \frac{4}{3}\right) = 15 + 8 = 23.$$