CHMMC Power Round 2019

November 17, 2019

1 Introduction (13 pts)

The goal of this power round is to answer the following question.

Question. For which positive integers n is it possible to split a square into n triangles of equal area?

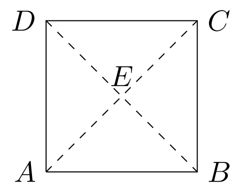

Figure 1: A square $ABCD$ is split into $n = 4$ triangles ΔABE, ΔBCE, ΔECD, ΔAED of equal area.

This problem was posed by Fred Richman in 1965 [3]. In 1968 [2], John Thomas provided a partial solution, and in 1970 [1], Paul Monsky solved it in its entirety. Monsky's clever proof combines combinatorics and number theory. Over the course of this power round, you will rediscover Monsky's proof.

/5 pts **Problem 1.** Let n be a positive even integer. Show that it is possible to split a square into n triangles of equal area.

This answers our question for even n. It is much more difficult to answer the question for odd n.

Suppose we find a way to split a square into n triangles of equal area for some odd positive integer n. The next problem demonstrates how this construction can be used to answer our question for all odd $k > n$.

/6 pts **Problem 2.** Suppose there is a way to split a square into n triangles of equal area for some odd positive integer n. Then there exists a way to split a square into $n + 2$ triangles of equal area for some positive integer n.

/2 pts **Problem 3.** Prove that exactly one of the following is true:

(1) For all positive odd integers n, it is not possible to split a square into n triangles of equal area.

(2) For all but finitely many positive odd integers n, it is possible to split a square into n triangles of equal area.

2 The case of $n = 3$ (14 pts)

In this section, we prove that it is impossible to split a square into three triangles of equal area.

/5 pts **Problem 4.** Let n, k be positive integers. If an n-gon is split into k triangles (not necessarily of equal area), then $k \geq n - 2$.

/9 pts **Problem 5.** Prove that it is not possible to split a square into three triangles of equal area. *(Hint: Consider where points of the triangles could lie and obtain a contradiction.)*

As you can see, the answer to our question for $n = 3$ is already quite involved. We need a different strategy for larger odd n. In the next two sections, we develop the necessary machinery to prove Monsky's Theorem.

3 The 2-adic magnitude (14 pts)

Definition 1. Let n be an integer. The **2-adic valuation** $v_2(n)$ is the largest integer e so that 2^e divides n.

Definition 2. Let a, b be integers with $b \neq 0$. Then we define

$$v_2\left(\frac{a}{b}\right) := v_2(a) - v_2(b).$$

Example 1.

- $v_2(4) = 2$.
- $v_2(3/4) = v_2(3) - v_2(4) = 0 - 2 = -2$. Moreover, $v_2(6/8) = v_2(6) - v_2(8) = 1 - 3 = -2$.

/3 pts **Problem 6.** Describe the set of all rational numbers n for which $v_2(n) = -1$.

Definition 3. For a rational number x, the **2-adic magnitude** $M_2(x)$ is given by

$$M_2(x) = \begin{cases} 2^{-v_2(x)} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Example 2.

- $M_2(4) = 2^{-2} = \frac{1}{4}$.
- $M_2(3/4) = M_2(6/8) = 2^{-2} = 4$.

/7 pts **Problem 7.** Let x, y be rational numbers. Then prove the following:

(a) $M_2(xy) = M_2(x)M_2(y)$.

(b) $M_2(x) = M_2(-x)$.

(c) If $M_2(x) \neq M_2(y)$, then $M_2(x + y) = \max\{M_2(x), M_2(y)\}$.

/4 pts **Problem 8.** For rational numbers x and y, given $M_2(x) > 1$ and $xy = 1$, show that $M_2(y) < 1$. Furthermore, show that if y is an integer then y is even.

4 Sperner's Lemma (15 pts)

Definition 4. Let P be a polygon. A **triangulation** is a subdivision of P into triangles.

Example 3. See Figure 2.

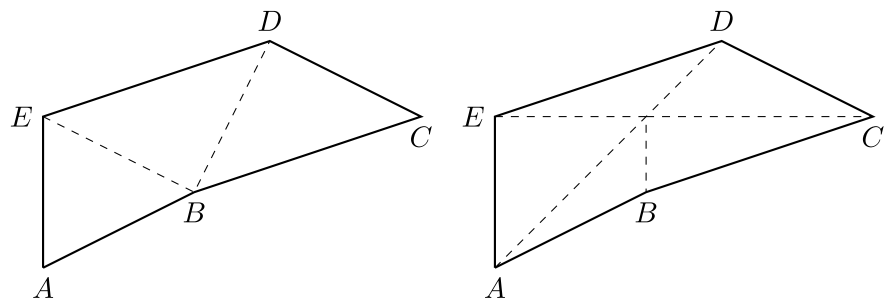

Figure 2: Two triangulations of $ABCDE$.

Definition 5.

1. Given a triangulation of a polygon, we can color each vertex of each triangle in the triangulation red (r), green (g), or blue (b). This is called a **rgb-coloring**.
2. Suppose the endpoints of an edge have colors c_1 and c_2. This edge is called a $\mathbf{c_1c_2}$**-edge**.
3. The **boundary** of the triangulated polygon is the collection of edges lying on the boundary of the original polygon.
4. A triangle is **complete** if it has one red vertex, one green vertex, and one blue vertex.

Example 4. See Figure 3.

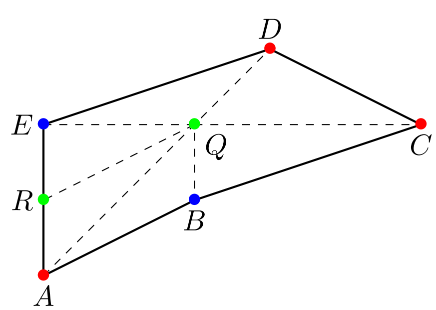

Figure 3: An rgb-coloring of a triangulation of $ABCDE$. The rg-edges are RA, QC, and QD. The boundary consists of the edges AB, BC, CD, DE, ER, and RA. The complete triangles are ΔABQ, ΔBCQ, and ΔQDE.

Lemma 1 (Sperner)**.** *Let P be a polygon, and consider any rgb-coloring of a triangulation of P. Let C_1 be the number of complete triangles in the triangulation. Let C_2 be the number of rg-edges on the boundary of the triangulated P. Then $C_1 - C_2$ is even.*

Example 5. Consider the rgb-coloring of the triangulated polygon in Figure 3. The complete triangles are ΔABQ, ΔBCQ, and ΔQDE, so $C_1 = 3$. The rg-edges on the boundary are RA, so $C_2 = 1$. Then $C_1 - C_2 = 2$ is even, as Sperner's Lemma predicts.

/15 pts **Problem 9.** Prove Sperner's Lemma. *Hint: Draw a dot on each side of a rg-edge, as in Figure 4. Count the number of dots on the inside and outside of P.*

Example 6. See Figure 4.

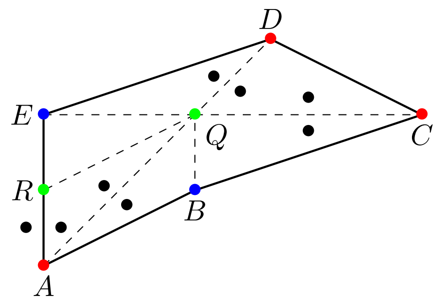

Figure 4: A dot is drawn on either side of each rg-edge.

5 Monsky's Theorem (34 pts)

In this section, you may assume that all points have rational coordinates. (Essentially the same proof provided here holds for all points in the plane.)

Definition 6. We define the following three sets of points in the plane.

$$\begin{aligned} S_1 &= \{(x, y) : \ M_2(x) < 1, \ M_2(y) < 1\} \\ S_2 &= \{(x, y) : \ M_2(x) \geq 1, \ M_2(x) \geq M_2(y)\} \\ S_3 &= \{(x, y) : \ M_2(y) \geq 1, \ M_2(y) > M_2(x)\} \end{aligned}$$

/4 pts **Problem 10.** Prove that S_1, S_2, S_3 form a **partition** of the plane. (This means that every point (x, y) lies in exactly one of the three sets S_1, S_2, S_3.)

/5 pts **Problem 11.** Let $(x_1, y_1) \in S_1$, $(x_2, y_2) \in S_2$ and $(x_3, y_3) \in S_3$. Then:

(a) We have $(-x_1, -y_1) \in S_1$.

(b) We have $(x_1 + x_2, y_1 + y_2) \in S_2$ and $(x_1 + x_3, y_1 + y_3) \in S_3$. (In other words, S_2 and S_3 are invariant under translation by elements in S_1.)

The following lemma will be useful for the next problem.

Lemma 2. *Consider a triangle with vertices* $(0,0)$*,* (a,b)*, and* (c,d)*. Then its area is* $|\frac{ad-bc}{2}|$*.*

/5 pts **Problem 12.** Let ΔABC be a triangle in the plane with $A \in S_1$, $B \in S_2$, and $C \in S_3$. Then $M_2(\text{Area}(\Delta ABC)) > 1$.

Finally, we are ready to prove Monsky's theorem.

Theorem 1. *Let* n *be an odd positive integer. It is not possible to split a square into* n *triangles of equal area.*

/20 pts **Problem 13.** Prove Monsky's theorem. *Hint: Without loss of generality, consider the square* $ABCD$ *with vertices* $A = (0,0)$*,* $B = (1,0)$*,* $C = (1,1)$*,* $D = (0,1)$*. Look at both rgb-colorings of* ABC *and the partition of* $ABCD$ *into* S_1*,* S_2*, and* S_3 *to show that* n *must be even.*

CHMMC 2019 Power Round written by Sara Fish (Caltech '21).

References

[1] P. Monsky, *On Dividing a Square Into Triangles*. The American Mathematical Monthly, Vol. 77, No. 2, (Feb 1970), pp. 161-164.

[2] J. Thomas, *A dissection problem*. Mathematics Magazine, Vol. 41, No. 4 (Sep 1968), pp. 187-190.

[3] M. Xu, *Sperner's Lemma*. Manuscript. URL: `https://math.berkeley.edu/~moorxu/oldsite/misc/equiareal.pdf`.