CMM 2025 Integration Bee Qualification Test Solutions

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Problem 1. $\int_0^\infty x^7 \exp(-x^2)\ \mathrm{d}x$

Proposed by Sam Murray

Solution: $\boxed{3}$

We let $u = x^2$ (and $\mathrm{d}u = 2x\ \mathrm{d}x$) to obtain the following.

$$\int_0^\infty x^6 \exp(-x^2) \cdot x\ \mathrm{d}x = \int_0^\infty \frac{u^3 \cdot \exp(-u)}{2}\ \mathrm{d}u$$

We now do tabular integration as follows, noting that $\exp(-\infty) \to 0$ and decays the polynomial terms.

$$\begin{aligned}
\int_0^\infty \frac{u^3 \cdot \exp(-u)}{2}\ \mathrm{d}u &= \frac{1}{2} \cdot \left[\exp(-u) \cdot (-u^3 - 3u^2 - 6u - 6)\right]_0^\infty \\
&= \frac{1}{2} \cdot \left[\exp(-u) \cdot (u^3 + 3u^2 + 6u + 6)\right]_0 \\
&= \frac{1}{2} \cdot [6 \cdot \exp(-0)] = \boxed{3}
\end{aligned}$$

Problem 2. $\int_0^\pi \sin^5(x)\ \mathrm{d}x$

Proposed by Sam Murray

Solution: $\boxed{\frac{8}{15}}$

We split $\sin^5(x)$ into a product to utilize the Pythagorean identity and utilize a u-substitution.

$$\begin{aligned}
\int_0^\pi \sin^4(x) \cdot \sin(x)\ \mathrm{d}x &= \int_0^\pi (1 - \cos^2(x))^2 \cdot \sin(x)\ \mathrm{d}x \\
&= \int_1^{-1} -(1 - u^2)^2\ \mathrm{d}u && (u = \cos(x)) \\
&= 2 \cdot \int_0^1 (1 - 2u^2 + u^4)\ \mathrm{d}u && \text{(even integrand)} \\
&= 2 \cdot \left[u - \frac{2u^3}{3} + \frac{u^5}{5}\right]_0^1 \\
&= 2 \cdot \left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{2(15 - 10 + 3)}{15} = \boxed{\frac{16}{15}}
\end{aligned}$$

Problem 3. $\int_{-3}^3 \frac{x^4}{1 + e^{-x}}\ \mathrm{d}x$

Proposed by Sam Murray

Solution: $\boxed{\dfrac{243}{5}}$

Recall King's rule as follows.

$$\int_a^b f(x)\ \mathrm{d}x = \int_a^b f(a+b-x)\ \mathrm{d}x$$

Since $a = 3, b = -3$, we have that $f(a+b-x) = f(-x)$. Adding these two versions of the integral (labeled $\mathscr{I}$), we obtain the following.

$$\begin{aligned}
2\mathscr{I} &= \int_{-3}^{3} \frac{x^4}{1+e^{-x}}\ \mathrm{d}x + \int_{-3}^{3} \frac{x^4}{1+e^{x}}\ \mathrm{d}x \\
&= \int_{-3}^{3} \frac{x^4 e^x}{e^x+1}\ \mathrm{d}x + \int_{-3}^{3} \frac{x^4}{1+e^{x}}\ \mathrm{d}x \\
&= \int_{-3}^{3} \frac{x^4(e^x+1)}{e^x+1}\ \mathrm{d}x \\
&= \int_{-3}^{3} x^4\ \mathrm{d}x = \left[\frac{x^5}{5}\right]_{-3}^{3} = \frac{243}{5}\cdot 2 \\
\mathscr{I} &= \boxed{\frac{243}{5}}
\end{aligned}$$

Problem 4. $\displaystyle\int_0^1 (\sin(\log x) + \cos(\log x))\ \mathrm{d}x$

Proposed by Sam Murray

Solution: $\boxed{0}$

One can let $x = e^u$ (and $\mathrm{d}x = e^u\ \mathrm{d}u$), and then perform integration by parts on the former term to simplify.

$$\begin{aligned}
\int_0^1 (\sin(\log x) + \cos(\log x))\ \mathrm{d}x &= \int_{-\infty}^{0} (\sin(u) + \cos(u)) \cdot e^u\ \mathrm{d}u \\
&= \int_{-\infty}^{0} e^u \sin(u)\ \mathrm{d}u + \int_{-\infty}^{0} e^u \cos(u)\ \mathrm{d}u \\
&= \left[e^u \sin(u) \Big|_{-\infty}^{0} - \int_{-\infty}^{0} e^u \cos(u)\ \mathrm{d}u \right] + \int_{-\infty}^{0} e^u \cos(u)\ \mathrm{d}u \\
&= e^0 \sin(0) = \boxed{0}
\end{aligned}$$

Problem 5. $\displaystyle\int_{-1}^{1} \frac{\cos(x)}{\arctan(x)}\ \mathrm{d}x$

Proposed by Sam Murray

Solution: $\boxed{0}$

The integrand is odd due to the presence of $\arctan(x)$. Since this is integrated across symmetric bounds, this integral immediately evaluates to $\boxed{0}$.

Problem 6. $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \sqrt{1+\cos(x)}\,\mathrm{d}x$

Proposed by Sam Murray

Solution: $\boxed{2\sqrt{2}-2}$

We can utilize the half-angle relation and integrate without the presence of the square root.

$$\begin{aligned}
\int_{\frac{\pi}{2}}^{\pi} \sqrt{1+\cos(x)}\,\mathrm{d}x &= \int_{\frac{\pi}{2}}^{\pi} \sqrt{2\cos^2\left(\frac{x}{2}\right)}\,\mathrm{d}x \\
&= \sqrt{2}\cdot\int_{\frac{\pi}{2}}^{\pi} \left|\cos\left(\frac{x}{2}\right)\right|\,\mathrm{d}x \\
&= \sqrt{2}\cdot\int_{\frac{\pi}{2}}^{\pi} \cos\left(\frac{x}{2}\right)\,\mathrm{d}x && (\cos(x)\geq 0 \text{ for } x\in[0,\pi/2]) \\
&= 2\sqrt{2}\cdot\sin\left(\frac{x}{2}\right)\Big|_{\frac{\pi}{2}}^{\pi} \\
&= 2\sqrt{2}\cdot\sin\left(\frac{\pi}{2}\right) - 2\sqrt{2}\cdot\sin\left(\frac{\pi}{4}\right) = \boxed{2-\sqrt{2}}
\end{aligned}$$

Problem 7. $\displaystyle\int_{-\infty}^{0} e^{-\sqrt[6]{-x}}\,\mathrm{d}x$

Proposed by Sam Murray

Solution: $\boxed{720}$

We do a u-substitution of $x=-u^6$ (and $\mathrm{d}x=-6u^5\ \mathrm{d}u$) to reduce the complexity of the exponent as follows, and follow with tabular integration to vanish the polynomial portion of the integrand.

$$\begin{aligned}
\int_{-\infty}^{0} e^{-\sqrt[6]{-x}}\,\mathrm{d}x &= \int_{\infty}^{0} e^{-|u|}\ \left(-6u^5\ \mathrm{d}u\right) \\
&= \int_{0}^{\infty} 6u^5\cdot e^{-u}\ \mathrm{d}u \\
&= \left[\exp(-u)\cdot(-6u^5-30u^4-120u^3-360u^2-720u-720)\right]_0^{\infty} \\
&= \left[\exp(-u)\cdot(6u^5+30u^4+120u^3+360u^2+720u+720)\right]_0 \\
&= [720\cdot\exp(-0)] = \boxed{720}
\end{aligned}$$

Problem 8. $\displaystyle\int_{1}^{\infty} \frac{1}{2025^{\lfloor x\rfloor}}\,\mathrm{d}x$

Proposed by Sam Murray

Solution: $\boxed{\dfrac{1}{2024}}$

We can re-write the integral as an infinite summation based on each interval it is on as follows.

$$\begin{aligned}\int_1^\infty \frac{1}{2025^{\lfloor x\rfloor}}\,\mathrm{d}x &= \sum_{k=1}^{\infty}\left(\int_k^{k+1}\frac{1}{2025^{\lfloor x\rfloor}}\,\mathrm{d}x\right)\\ &= \sum_{k=1}^{\infty}\left(\int_k^{k+1}\frac{1}{2025^{k}}\,\mathrm{d}x\right)\\ &= \sum_{k=1}^{\infty}\frac{1}{2025^k}\\ &= \frac{\frac{1}{2025}}{1-\frac{1}{2025}} = \boxed{\frac{1}{2024}}\end{aligned}$$

Problem 9. $\displaystyle\int_4^5 \frac{2x+1}{(x-1)(x-2)(x-3)}\,\mathrm{d}x$

Proposed by Sam Murray

Solution: $\boxed{\dfrac{23\ln(2)-13\ln(3)}{2}}$

We note that the integrand will admit a partial fraction decomposition as follows.

$$\begin{aligned}\frac{2x+1}{(x-1)(x-2)(x-3)} &= \frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}\\ 2x+1 &= A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)\\ x=1 &\implies 3=A(-1)(-2)\implies A=\frac{3}{2}\\ x=2 &\implies 5=B(1)(-1)\implies B=-5\\ x=3 &\implies 7=C(1)(2)\implies C=\frac{7}{2}\end{aligned}$$

Therefore, we can integrate the partial fraction decomposition as follows.

$$\begin{aligned}\int_4^5 \frac{2x+1}{(x-1)(x-2)(x-3)}\,\mathrm{d}x &= \int_4^5 \frac{3}{2(x-1)}-\frac{5}{x-2}+\frac{7}{2(x-3)}\,\mathrm{d}x\\ &= \left[\frac{3}{2}\ln(x-1)-5\ln(x-2)+\frac{7}{2}\ln(x-3)\right]_4^5\\ &= \left[\frac{3}{2}\ln(4)-5\ln(3)+\frac{7}{2}\ln(2)\right]-\left[\frac{3}{2}\ln(3)-5\ln(2)+\frac{7}{2}\ln(1)\right]\\ &= \left[3\ln(2)-5\ln(3)+\frac{7}{2}\ln(2)\right]-\left[\frac{3}{2}\ln(3)-5\ln(2)\right]\\ &= \boxed{\frac{23\ln(2)-13\ln(3)}{2}}\end{aligned}$$

Problem 10. $\int_0^1 xe^{x^2+e^{x^2}}\ \mathrm{d}x$

Proposed by Sam Murray

Solution: $\boxed{\dfrac{e^e-e}{2}}$

One can recognize that taking the derivative of the following yields an expression similar to our integrand. We can then integrate both sides to obtain the desired result.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{e^{x^2}}\right) &= e^{e^{x^2}}\cdot\left(e^{x^2}\right)' \\
&= e^{e^{x^2}}\cdot e^{x^2}\cdot 2x \\
\int_0^1 2\cdot xe^{x^2+e^{x^2}}\ \mathrm{dx} &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{e^{x^2}}\right)\ \mathrm{d}x \\
\int_0^1 xe^{x^2+e^{x^2}}\ \mathrm{dx} &= \frac{1}{2}\cdot e^{e^{x^2}}\Big|_0^1 = \boxed{\frac{e^e-e}{2}}
\end{aligned}$$

Problem 11. $\int_0^{\frac{3}{2}} \lfloor x+\lfloor 2x\rfloor\rfloor\ \mathrm{d}x$

Proposed by Sam Murray

Solution: $\boxed{2}$

We can use the $\lfloor 2x\rfloor$ to split up the integral as follows.

$$\begin{aligned}
\int_0^{\frac{3}{2}} \lfloor x+\lfloor 2x\rfloor\rfloor\ \mathrm{d}x &= \int_0^{\frac{1}{2}} \lfloor x+\lfloor 2x\rfloor\rfloor\ \mathrm{d}x + \int_{\frac{1}{2}}^1 \lfloor x+\lfloor 2x\rfloor\rfloor\ \mathrm{d}x + \int_1^{\frac{3}{2}} \lfloor x+\lfloor 2x\rfloor\rfloor\ \mathrm{d}x \\
&= \int_0^{\frac{1}{2}} \lfloor x\rfloor\ \mathrm{d}x + \int_{\frac{1}{2}}^1 \lfloor x+1\rfloor\ \mathrm{d}x + \int_1^{\frac{3}{2}} \lfloor x+2\rfloor\ \mathrm{d}x \\
&= \int_{\frac{1}{2}}^1 1\ \mathrm{d}x + \int_1^{\frac{3}{2}} 3\ \mathrm{d}x = \frac{1}{2}+\frac{3}{2} = \boxed{2}
\end{aligned}$$

Problem 12. $\int_{2025}^{2024} \ln(\{x\})\ \mathrm{d}x$

Proposed by Sam Murray

Solution: $\boxed{1}$

Recall that $\{x\} = x - \lfloor x \rfloor$, of which the latter term is constant on the interval as follows.

$$\begin{aligned}\int_{2025}^{2024} \ln(x - \lfloor x \rfloor)\ \mathrm{d}x &= \int_{2025}^{2024} \ln(x - 2024)\ \mathrm{d}x \\ &= \int_{1}^{0} \ln(u)\ \mathrm{d}u \\ &= [u \ln(u) - u]_1^0 \\ &= \boxed{1}\end{aligned}$$

Problem 13. $\int_{98}^{99} x^{-\frac{2}{\log x}}\ \mathrm{d}x$

Proposed by Sam Murray

Solution: $\boxed{e^{-2}}$

Upon letting $x = e^u$, we drastically simplify the integral and proceed normally.

$$\begin{aligned}\int_{98}^{99} x^{-\frac{2}{\log x}}\ \mathrm{d}x &= \int_{\ln(98)}^{\ln(99)} (e^u)^{-\frac{2}{\log e^u}}\ (e^u\ \mathrm{d}u) \\ &= \int_{\ln(98)}^{\ln(99)} (e^u)^{-\frac{2}{u}}\ (e^u\ \mathrm{d}u) \\ &= \int_{\ln(98)}^{\ln(99)} \frac{e^u}{e^2}\ \mathrm{d}u \\ &= \frac{e^u}{e^2}\bigg|_{\ln(98)}^{\ln(99)} = \frac{99-98}{e^2} = \boxed{e^{-2}}\end{aligned}$$

Problem 14. $\int_{-\infty}^{\infty} e^{-x^2+3x}\ \mathrm{d}x$

Proposed by

Solution: $\boxed{e^{9/4}\sqrt{\pi}}$

We note that all densities integrate to 1, and the standard normal density allows us to make the following relation.

$$\begin{aligned}1 &= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{-x^2/2}\ \mathrm{d}x \\ \sqrt{2\pi} &= \int_{-\infty}^{\infty} \exp(-x^2/2)\ \mathrm{d}x \\ \sqrt{2\pi} &= \sqrt{2} \cdot \int_{-\infty}^{\infty} \exp(-u^2)\ \mathrm{d}u && (x = \sqrt{2}u, \mathrm{d}x = \sqrt{2}\ \mathrm{du}) \\ \sqrt{\pi} &= \int_{-\infty}^{\infty} \exp(-u^2)\ \mathrm{d}u\end{aligned}$$

Therefore, it suffices to complete the square in the desired integral since the above is translation invariant.

$$\begin{aligned}
\int_{-\infty}^{\infty} \exp(-x^2+3x)\ \mathrm{d}x &= \int_{-\infty}^{\infty} \exp\left(-x^2+3x-\frac{9}{4}+\frac{9}{4}\right)\ \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \exp\left(-\left(x-\frac{3}{2}\right)^2+\frac{9}{4}\right)\ \mathrm{d}x \\
&= e^{9/4}\cdot\int_{-\infty}^{\infty} \exp\left(-u^2\right)\ \mathrm{d}u & \left(u = x-\frac{3}{2}, \mathrm{d}u = \mathrm{d}x\right) \\
&= \boxed{e^{9/4}\cdot\sqrt{\pi}}
\end{aligned}$$