Individual Round Solutions

2014 CHMMC

Solution 1. The sum of rows equals the sum of columns, so $n = \boxed{46}$.

Solution 2. The maximum occurs when as many categories as possible are present. Since there are only 14 socks present, we can have only 14 categories, so one color/length combination is missing. If we had all 15 categories, we could pick any of the 5*3 categories and any of the 4*2 distinct categories to give 5*4*3*2/2 = 60 possibilities, but since one is missing, we must discount $4 * 2 = 8$ of these, giving $\boxed{52}$.

Solution 3. The prime factorization of 2014 is $2 * 19 * 53$, so groups can only come in the sizes 1, 2, 19, 38, 53, 106, 1007, 2014. Clearly 1, 2, 1007, 2014 require at least half the votes to win, so ignore those. Let n be the number of groups and $m = 2014/n$ be the group size. It requires fewer votes to win a group than to prevent the opponent from winning two groups since $\lceil 2m/3 \rceil \leq 2(\lfloor 1 + m/3 \rfloor)$. This means that the minimum number of votes is

$$V = \lceil 2m/3 \rceil (\lfloor 1 + n/2 \rfloor)$$

which minimizes when $n = \boxed{19}$ by trying the possibilities.

Solution 4. We can see that his utility maximizes on a day when he gains utility but would lose utility if he waited another day. Waiting another day means his (positive) utility gets multiplied by $103/101 * (364 - k)/(365 - k)$, meaning that he should stop when (letting $l = 365 - k$) $(l - 1)/l < 101/103$. This gives $2/103l < 1$, so $l <= 51$. Therefore he should sell on day $365 - 51 = \boxed{314}$. Fun fact: when he sells, he has more than 472 times as many sheep as he started with.

Solution 5. Drawing a line $\overline{BC}$ we can see that by law of cosines $\overline{BC} = \sqrt{\frac{27}{2}}$. Because the tangents are angle $\frac{\pi}{3}$ apart, the radii of the arc that connect to the tangents must be $\frac{2\pi}{3}$ apart. Thus, by law of cosines, and knowing that all radii are equal, the radius of the arc is 3. The measure of the arc must be 2π minus the angle between the radii to the endpoints. Thus, the measure of the arc is $\frac{4\pi}{3}$. Thus, the length of $\widehat{BC}$ is $\boxed{4\pi}$.

Note: due to an ambiguity in the problem, $\boxed{2\pi}$ was also an acceptable answer.

Solution 6. If we consider this mod 8, we can only ever have values of 0, 1, or 4. 0 can change to either 0 or 1, 1 can change to 1 or 4, and 4 can change to 0 in two ways. Therefore we can only cycle through 0 to 1 to 4 to 0 at most once, so we have to stay at 0 or 1 at least once. There are 2 ways to do this by staying at 1, 4 ways to do this by staying at 0 once, and 1 way to do this by staying at 0. This gives $\boxed{7}$ ways. Alternatively, look at the fourth power of the adjacency matrix where 0 is the first vertex, 1 is the second, and 4 is the third:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}^4 = \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}^2 \right)^2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}^4 = \left(\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \right)^2$$

The top left corner of this is the desired quantity, 7. Maybe the easiest solution is to make a little diagram with the numbers 0, 1, 4 repeated 5 times and count the paths that start at the top row's 0 and end at the bottom row's 0. There are actually a lot of ways to count these sequences. Try making a recurrence relation and finding how many sequences of length 20 there are!t

Solution 7. After k repetitions of the shuffling procedure, the card that was originally in position i, counting from the top of the deck, will be in position $i2^k$, so the smallest n that returns the deck to the original state is the smallest n with $2^n \equiv 1 \pmod 9$, which is $\boxed{6}$.

Solution 8. Let O be the center of the circle, and draw the perpendicular P from O to AC. Then CPO and APO are congruent, meaning the length of BP is $(1+7)/2-1=3$. Also notice that the measure of angle POB is $45°$, so ΔPOB is isosceles; the length of OP is 3. Since CP's length is 4, we have by the Pythagorean theorem that the circle's radius is $\boxed{5}$.

Solution 9. Let g_n denote z^n+1/z^n. Expanding shows that $g_{n+m}=g_ng_m-g_{|n-m|}$, so in particular

$$g_{2^n+2^{n-1}}=g_{2^n}g_{2^{n-1}}-g_{2^{n-2}}$$

And

$$g_{2^{n+1}}=g_{2^n}^2-2$$

Using this second relation, we can see that since $g_1=1$, for all $n\geq 1$ we have $g_{2^n}=-1$. Using the second relation then gives that

$$g_{2^n+2^{n-1}}=1--1$$

$$g_{2^n+2^{n-1}}=2$$

Since 96 is 2^6+2^5, it takes this form, so the answer is $\boxed{2}$.

Solution 10. $\boxed{\frac{2}{3}}$. First, we know that

$$\tan(a-b)=\frac{\tan a-\tan b}{1+\tan a\tan b}=\frac{4}{\frac{1}{\tan b}+5\tan b}.$$

In addition, since y is acute, $\tan y>0$.

So we can use $1/x+x\geq 2$ to see that the denominator is at least $2\sqrt{5}$, and hence the expression is at most $2/\sqrt{5}$. Constructing a right triangle with this tangent shows that it has a sin of $\boxed{2/3}$.

Solution 11. Let $\overline{SM}/\overline{SB}$ be a ratio x, and $\overline{SN}/\overline{SD}$ be a ratio y. The region $SAMPN$ can be split in half two ways: into either $SAMP$ and $SANP$ or $SAMN$ and $SPMN$. Thus

$$V(SAMP)+V(SANP)=V(SAMN)+V(SPMN)$$

Dividing by the volume of $V(SABC)$, which is equal to $V(SBCD),V(SCDA)$, or $V(SDAB)$ since the base is a parallelogram:

$$\frac{V(SAMP)}{V(SABC)}+\frac{V(SANP)}{V(SDCA)}=\frac{V(SAMN)}{V(SDAB)}+\frac{V(SPMN)}{V(SBCD)}$$

If we consider SBC to be the base of $SABC$, we see that since SBP has half the area of the base and the same height, $\frac{V(SABP)}{V(SABC)}=\frac{1}{2}$. An analogous argument shows that $\frac{V(SAMP)}{V(SABC)}=\frac{1}{2}\overline{SM}/\overline{SB}=\frac{x}{2}$. Applying this to all sides gives

$$\frac{x}{2}+\frac{y}{2}=xy+\frac{1}{2}xy$$

$$x+y=3xy$$

$$y=\frac{x}{-1+3x}$$

And since $V(SABCD)=2V(SABC)=2V(SDCA)$, we want to maximize

$$\frac{V(SAMP)}{2V(SABC)}+\frac{V(SANP)}{2V(SDCA)}=\frac{x}{4}+\frac{y}{4}$$

$$=\frac{3x^2}{4(3x-1)}$$

$$=\frac{1}{12}((3x-1)+2+\frac{1}{3x-1})$$

where x is between $1/2$ and 1. This minimizes when $3x-1=1$, giving $x=2/3$ and a ratio of $r_1=\frac{1}{3}$. It maximizes when $x=1$, giving a ratio of $r_2=\frac{3}{8}$. Thus the solution is $\boxed{\left(\frac{1}{3},\frac{3}{8}\right)}$.

Solution 12. Consider a full 5x5 grid and define the default "permutation" of pieces to be one in which all pieces are on the missing diagonal. Let the ith pieces of the board be the piece on the ith column, so that any permutation just moves each piece into a different row. Let $A_{i_1 \ldots i_k}$ be the set of permutations which keep pieces $i_1, \ldots i_k$ on the main missing diagonal. For instance, A_1 is the set of permutations which keeps the first piece in the corner, and A_{14} is the set of permutations where the 1st and 4th pieces remain on the 1st and 4th rows respectively. Let B be the set of permutations which do not leave any pieces on the missing diagonal; that is, $B = A \setminus A_1 \setminus \ldots A_5$. By the inclusion-exclusion principle, we have

$$|B| = |A| - |A_1| - \ldots - |A_5| + |A_{12}| + \ldots |A_{45}| - \ldots + \ldots - \ldots |A_{12345}|$$

That is, we have an expression for the number of permutations which do not leave any of the pieces in their original places. Since in $A_{i_1,\ldots i_k}$ we have freedom to move $5-k$ pieces, we have $|A_{i_1,\ldots i_k} = (5-k)!$. Since there are $\binom{5}{k}$ ways to pick this, we have

$$|B| = 120 - 5(24) + 10(6) - 10(2) + 5(1) - 1$$

$$|B| = \boxed{44}$$

Solution 13. Suppose the country has roads between any pair of cities, and we are trying to destroy certain roads so that no such cycle (a Hamiltonian cycle) exists anymore. The most efficient way to do this is to destroy all but one road from a particular city; this destroys 18 roads, leaving us with $\binom{20}{2} - 18 = \boxed{172}$. We can see this because we want to destroy roads which share the fewest number of Hamiltonian cycles with each other as possible; each road is originally a part of the same number of cycles, and each demolition is only good for destroying those cycles which no other demolition destroyed. In the complete country, roads from the same city share only 1/18 of their cycles, while roads from different pairs of cities share 2/19 of their cycles by symmetry. While not completely rigorous (the simplest proof I know uses Chvatal's condition), the intuition is what matters in this problem.

Solution 14. Use Vieta's to extract the semiperimeter of the quadrilateral, which is $10/2 = 5$, then plug that into the quartic to get 19. By Brahmagupta's formula, we have the area is $(s - r_1)(s - r_2)(s - r_3)(s - r_4) = 19$ where the r_i's are the roots. Thus the area is the square root of that, or $\boxed{\sqrt{19}}$.

Solution 15. Notice that if $u_{n,m}$ depends only on $n+m$, then $u(n-1+m)+u(n+1+m) = u(n+m-1)+u(n+m+1)$, satisfying the equation. Similarly, if $u_{n,m}$ depends only on $n - m$, then $u(n-1-m) + u(u+1-m) = u(n-m+1) + u(n-m-1)$, satisfying the equation again. Since the equation is linear, any sum of two such functions f and g would satisfy the equation. We can do this easily: if $f(m) + g(-m) = m^2 + m$ and $f(1+m) + g(1-m) = m^3 + 3m$, then $f(n+m) = (m+n)^2$ and $g(n-m) = m-n$, so that the equation is satisfied. Thus $u_{30,-5} = 25^2 - 5 - 30 = \boxed{590}$.