CHMMC
Caltech Harvey Mudd Math Competition

Individual Round 2022-2023 Solutions

Problem 1. Given any four digit number $X = \underline{ABCD}$, consider the quantity $Y(X) = 2 \cdot \underline{AB} + \underline{CD}$. For example, if $X = 1234$, then $Y(X) = 2 \cdot 12 + 34 = 58$. Find the sum of all natural numbers $n \leq 10000$ such that over all four digit numbers X, the number n divides X if and only if it also divides $Y(X)$.

Proposed by Natalie Couch

Solution: $\boxed{10171}$.

For $X = \underline{ABCD}$ note that $X - Y(X) = 100 \cdot \underline{AB} + \underline{CD} - (2 \cdot \underline{AB} + \underline{CD}) = 98 \cdot \underline{AB}$. Hence, any positive divisor n of 98, i.e. $n \in \{1, 2, 7, 14, 49, 98\}$, divides X if and only if it divides $Y(X)$. Moreover, the number $n = 10000$ is greater than X or $Y(X)$, hence does not divide either X or $Y(X)$, for any $X = \underline{ABCD}$. We claim these are all possible values of n, so that the answer is $1 + 2 + 7 + 14 + 49 + 98 + 10000 = \boxed{10171}$.

Given any natural number $n \leq 10000$ other than the ones listed above, we shall give $X = \underline{ABCD}$ such that n divides X but not $Y(X)$. For $300 \leq n < 10000$ this is immediate: let X be any multiple of n, and note n does not divide $Y(X)$ as $Y(X)$ is bounded above by $2 \cdot 99 + 99 = 297$. Similarly, for $124 \leq n < 300$, pick the smallest four digit number $X = \underline{ABCD}$ that is a multiple of n; then, $X \leq 1299$, so n does not divide $Y(X)$ as $Y(X)$ is bounded above by $2 \cdot 12 + 99 = 123$. For $100 \leq n < 123$, the number $X = 10n$ works by direct inspection. Finally, for $n < 100$ that does not divide 98, observe any sequence of 100 consecutive integers contains a divisor of n. Choose digits $A \geq 1, B$ such that $\underline{AB} \equiv 1 \pmod{n}$ or $\underline{AB} \equiv -1 \pmod{n}$, and there exist two digits C, D such that n divides $X := \underline{ABCD}$. Since n is coprime to $\underline{AB}$, n does not divide $98 \cdot \underline{AB}$, so n does not divide $Y(X)$.

Problem 2. A sink has a red faucet, a blue faucet, and a drain. The two faucets release water into the sink at constant but different rates when turned on, and the drain removes water from the sink at a constant rate when opened. It takes 5 minutes to fill the sink (from empty to full) when the drain is open and only the red faucet is on, it takes 10 minutes to fill the sink when the drain is open and only the blue faucet is on, and it takes 15 seconds to fill the sink when both faucets are on and the drain is closed. Suppose that the sink is currently one-thirds full of water, and the drain is opened. Rounded to the nearest integer, how many seconds will elapse before the sink is emptied (keeping the two faucets closed)?

Proposed by Brian Yang

Solution: $\boxed{11}$.

Let R_1, R_2 be the rates, in units/second, at which water is added to the sink due to the red and blue faucets, respectively, and let r be the rate, in units/second, at which water is removed from the sink due to the drain. WLOG assume the capacity of the sink is 600 units. Then, the problem conditions say

$$(R_1 - r)300 = (R_2 - r)600 = (R_1 + R_2)15 = 600.$$

Solving this system yields $r = \frac{37}{2}, R_1 = \frac{41}{2}, R_2 = \frac{39}{2}$. Thus, it takes $\frac{200}{37/2} = 10.\overline{810} \approx \boxed{11}$ seconds to empty the $\frac{1}{3}$ full sink with the drain.

Problem 3. One of the bases of a right triangular prism is a triangle XYZ with side lengths $XY = 13, YZ = 14, ZX = 15$. Suppose that a sphere may be positioned to touch each of the five faces of the prism at exactly one point. A plane parallel to the rectangular face of the prism containing $\overline{YZ}$ cuts the prism and the sphere, giving rise to a cross-section of area A for the prism and area 15π for the sphere. Find the sum of all possible values of A.

Proposed by Brian Yang

Solution: $\boxed{\frac{448}{3}}$.

For any sphere tangent to the three rectangular faces of the prism, its cross section determined by the three tangency points is its great circle, the radius of which is the inradius r of $\triangle XYZ$. Note that $[XYZ] = 84$: the X-altitude of $\triangle XYZ$ is of length 12 and it splits $\overline{YZ}$ into two segments of length 5 and 9. Then, applying $[XYZ] = rs$, where $s = 21$ is the semiperimeter of $\triangle XYZ$, we have $r = 4$. Hence, the height of the prism is 8.

Let $\mathscr{P}$ be the plane of the cut described above. Note $\mathscr{P}$ cuts the prism into two pieces, one of which is a triangular prism whose triangular base is similar to $\triangle XYZ$. If $0 \leq d \leq 4$ is the distance between $\mathscr{P}$ and the center O of the sphere, then the spherical cross-section has radius $\sqrt{16-d^2}$, hence area $\pi(16-d^2)$. Then, $\pi(16-d^2) = 15\pi$, so $d = 1$. Since $\triangle XYZ$ has inradius $r = 4$ and X-altitude 12, the distance between $\mathscr{P}$ and X is either 7 or 9. By similarity, altitudes of 7 and 9 correspond to bases of lengths $\frac{49}{6}$ and $\frac{21}{2}$. This yields rectangular cross sections of dimensions $\frac{49}{6} \times 8$ and $\frac{21}{2} \times 8$, respectively, so the answer is $8(\frac{49}{6} + \frac{21}{2}) = \boxed{\frac{448}{3}}$.

Problem 4. Albert, Brian, and Christine are hanging out by a magical tree. This tree gives each of them a stick, each of which have a non-negative real length. Say that Albert gets a branch of length x, Brian a branch of length y, and Christine a branch of length z, and the lengths follow the condition that $x+y+z = 2$.

Let m and n be the minimum and maximum possible values of $xy+yz+xz-xyz$, respectively. What is $m+n$?

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\frac{28}{27}}$.

Let $S = xy+yz+zx-xyz$. Notice that

$$P := (1-x)(1-y)(1-z) = 1-(x+y+z)+S = S-1.$$

so it is enough to find the minimum and maximum of P. There are three cases: all three terms $1-x, 1-y, 1-z$ are positive, in which $P > 0$, one or more terms $1-x, 1-y, 1-z$ are 0, in which $P = 0$, or one of the terms $1-x, 1-y, 1-z$ are negative, in which the other two are positive and so $P < 0$.

Suppose all three terms $1-x, 1-y, 1-z$ are positive, i.e. $x, y, z < 1$. Under these constraints, the AM-GM inequality reads

$$\sqrt[3]{P} = \sqrt[3]{(1-x)(1-y)(1-z)} \leq \frac{(1-x)+(1-y)+(1-z)}{3} = \frac{1}{3},$$

which implies $P \leq \frac{1}{27}$, with equality holding when $x = y = z = \frac{2}{3}$.

On the other hand, assume $1-x < 0$ (WLOG). Since $0 \leq x, y, z \leq 2$, we have $|1-x|, |1-y|, |1-z| \leq 1$. Thus, $|P| \leq 1$, in particular $P \geq -1$. Equality holds when $x = 2, y = 0, z = 0$.

It follows $m = -1+1 = 0, n = \frac{1}{27}+1 = \frac{28}{27}$, so the answer is $\boxed{\frac{28}{27}}$.

Problem 5. Let $\mathscr{S} := MATHEMATICSMATHEMATICSMATHE\ldots$ be the sequence where 7 copies of the word $MATHEMATICS$ are concatenated together. How many ways are there to delete all but five letters of $\mathscr{S}$ such that the resulting subsequence is $CHMMC$?

Proposed by Jeck Lim

Solution: $\boxed{434}$.

For any subsequence $CHMMC$ in $\mathscr{S}$ let $1 \le i_1, i_2, \ldots, i_5 \le 7$ be indices that denote the copy of $MATHEMATICS$ in which C, H, M, M, C are contained in, respectively. For convenience, let M_1, M_2 be the first and second M in $CHMMC$. Each copy of $MATHEMATICS$ contains a unique C and H, while each copy of $MATHEMATICS$ contains two M's, so we only need do casework on the 4 possible cases of the positions of M_1, M_2 relative to their respective copies of $MATHEMATICS$: they are either both the first M, both the second M, or one of them is the first M and the other the second M. Counting the possible choices of indices in each case, we have

- Case 1: M_1, M_2 are the first M's. In this case we observe that $1 \le i_1 < i_2 < i_3 < i_4 \le i_5 \le 7$ yielding $\binom{8}{5}$ choices of indices.
- Case 2: M_1 is the first M, M_2 is the second M. In this case we observe that $1 \le i_1 < i_2 < i_3 \le i_4 \le i_5 \le 7$ yielding $\binom{9}{5}$ choices of indices.
- Cases 3 and 4: M_1 is the second M. In this case we observe that $1 \le i_1 < i_2 \le i_3 < i_4 \le i_5 \le 7$, regardless of whether M_2 is the first or second M, yielding $\binom{9}{5}$ choices of indices for each case.

Thus, there are $\binom{8}{5} + \binom{9}{5} + \binom{9}{5} + \binom{9}{5} = \boxed{434}$ such subsequences.

Problem 6. Consider two sequences of integers a_n and b_n such that $a_1 = a_2 = 1,\ b_1 = b_2 = 1$ and that the following recursive relations are satisfied for integers $n > 2$:

$$\begin{aligned} a_n &= a_{n-1}a_{n-2} - b_{n-1}b_{n-2}, \\ b_n &= b_{n-1}a_{n-2} + a_{n-1}b_{n-2}. \end{aligned}$$

Determine the value of

$$\sum_{1 \le n \le 2023, b_n \ne 0} \frac{a_n}{b_n}.$$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{675}$.

For any integer $n \ge 1$, set $z_n = a_n + b_n i \in \mathbb{C}$. Hence, $z_1 = z_2 = 1 + i$. For $n > 2$ notice that

$$\begin{aligned} z_n = a_n + b_n i &= (a_{n-1}a_{n-2} - b_{n-1}b_{n-2}) + (b_{n-1}a_{n-2} + a_{n-1}b_{n-2})i \\ &= (a_{n-1} + b_{n-1}i)(a_{n-2} + b_{n-2}i) = z_{n-1}z_{n-2}. \end{aligned}$$

Thus, $z_3 = z_1 z_2 = (1+i)^2, z_4 = z_2 z_3 = (1+i)^3, \ldots$. Suppose $F_1, \ldots, F_{n-1}$ are non-negative integers such that $z_1 = (1+i)^{F_1}, z_2 = (1+i)^{F_2}, \ldots, z_{n-1}(1+i)^{F_{n-1}}$. Then, $z_n = z_{n-1}z_{n-2} = (1+i)^{F_{n-1}}(1+i)^{F_{n-2}} = (1+i)^{F_{n-1}+F_{n-2}}$. Therefore, $z_n = (1+i)^{F_n}$, where $\{F_n\}$ is a sequence of numbers given by $F_0 = 0, F_1 = F_2 = 1$ and the recursive relation $F_n = F_{n-1} + F_{n-2}$. In particular, F_n is the usual nth Fibonacci number.

Note $(1+i)^4 = -4 \in \mathbb{R}$, so the ratio $\mathrm{Re}(z_n) : \mathrm{Im}(z_n) = a_n : b_n$ depends only on F_n modulo 4. Since $F_5 \equiv 1 \pmod 4$, $F_6 = 8$, the residue of $\{F_n\}$ modulo 4 is the 6-periodic sequence $F_1, F_2, F_3, F_4, F_5, F_0, F_1, \ldots$. Thus, the sequence of ratios $\frac{a_n}{b_n}$ is a 6-periodic sequence whose first five terms (corresponding to $F_1, \ldots, F_5$) are $1, 1, 0, -1, 1$ (note $b_6 = 0$, so the ratio is "undefined" in that case). We conclude

$$\sum_{1 \le n \le 2023,\ b_n \ne 0} \frac{a_n}{b_n} = \frac{a_{2023}}{b_{2023}} + \frac{2022}{6} \cdot \left(\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \frac{a_4}{b_4} + \frac{a_5}{b_5} \right) = \boxed{675}.$$

Problem 7. Suppose ABC is a triangle with circumcenter O. Let A' be the reflection of A across $\overline{BC}$. If $BC = 12$, $\angle BAC = 60^\circ$, and the perimeter of ABC is 30, then find $A'O$.

Proposed by Jeff Ren

Solution: $\boxed{6\sqrt{3}}$.

Note that $\angle BOC = 120^\circ, \angle BA'C = 60^\circ$ so $A'BOC$ is cyclic. Since reflection preserves lengths and angles, $\triangle A'BC \cong \triangle ABC$, so the circumradius R of $A'BOC$ is the same as that of $\triangle ABC$, Applying Ptolemy's theorem to $A'BOC$, we have

$$A'O \cdot BC = A'B \cdot OC + BO \cdot CA' = R(A'B + CA') = R(AB + CA),$$

so $A'O = \frac{R(AB+CA)}{BC}$. By the law of sines, $\frac{BC}{R} = 2\sin\angle BAC = \sqrt{3}$, so $AO' = \frac{AB+CA}{\sqrt{3}} = \frac{18}{\sqrt{3}} = \boxed{6\sqrt{3}}$.

Problem 8. A class of 10 students wants to determine the class president by drawing slips of paper from a box. One of the students, Bob, puts a slip of paper with his name into the box. Each other student has a $\frac{1}{2}$ probability of putting a slip of paper with their own name into the box and a $\frac{1}{2}$ probability of not doing so. Later, one slip is randomly selected from the box. Given that Bob's slip is selected, find the expected number of slips of paper in the box before the slip is selected.

Proposed by Jeff Ren

Solution: $\boxed{\dfrac{5120}{1023}}$.

There is a $\frac{1}{2^9}\binom{9}{n-1}$ chance of a box with n slips and a $\frac{1}{n}$ chance of selecting Bob's slip thereafter. Hence, the probability there are n slips in the box and Bob's slip is selected thereafter is $\frac{1}{2^9 n}\binom{9}{n-1}$, and the overall probability Bob's slip is selected is $\sum_{n=1}^{10} \frac{1}{2^9 n}\binom{9}{n-1}$. Noting that $\frac{1}{n}\binom{9}{n-1} = \frac{1}{10}\binom{10}{n}$, the conditional expectation value is

$$\frac{\sum_{n=1}^{10} n \cdot \frac{1}{2^9 n}\binom{9}{n-1}}{\sum_{n=1}^{10} \frac{1}{2^9 n}\binom{9}{n-1}} = \frac{2^9}{\sum_{n=1}^{10} \frac{1}{10}\binom{10}{n}} = \frac{10 \cdot 2^9}{2^{10}-1} = \boxed{\dfrac{5120}{1023}}.$$

Problem 9. Let a and b be positive integers, $a > b$, such that $6! \cdot 11$ divides $x^a - x^b$ for all positive integers x. What is the minimum possible value of $a + b$?

Proposed by Ritvik Teegavarapu

Solution: $\boxed{68}$.

Notice $6! \cdot 11 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$. When $x = 2$, we have

$$6! \cdot 11 \mid 2^b(2^{a-b} - 1) \implies 2^4 \mid 2^b$$

since $2^{a-b} - 1$ is always odd. Thus, we must have $b \geq 4$ for the condition of the problem to hold.

By the CRT, the condition of the problem is equivalent to $x^a - x^b \equiv 0 \pmod{q}$ for all integers x, over $q = 2^4, 3^2, 5, 11$. If x is a multiple of 2 (resp. $3, 5, 11$), then the assertion $a \geq b > 4$ implies that $x^a - x^b \equiv 0 \pmod{q}$ for $q = 2^4$ (resp. $3^2, 6, 11$), so in what follows assume that x is relatively prime to q. Then, the statement $x^a - x^b \equiv 0 \pmod{q}$ is equivalent to $x^{a-b} \equiv 1 \pmod{q}$.

For $x \in (\mathbb{Z}/q\mathbb{Z})^*$ (the set of all residues of q relatively prime to q), denote by $|x|$ its multiplicative order modulo q. The least common multiple of the set of positive integers $\{|x| : x \in (\mathbb{Z}/q\mathbb{Z})^*\}$ is precisely $4, 6, 4, 10$ for $q = 2^4, 3^2, 5, 11$, respectively. That is to say, $x^{a-b} \equiv 1 \pmod{q}$ for all $x \in (\mathbb{Z}/q\mathbb{Z})^*$ if and only if $4, 6, 4, 10$

divide $a-b$ for $q=2^4,3^2,5,11$, respectively[1]. Hence, $a>b\geq 4$ are positive integers satisfying the problem condition if and only if $\mathrm{lcm}(4,6,10)$ divides $a-b$. Thus, $a-b=60k$ for some positive integer k. It is clear $a+b$ is minimized when $k=1$, attaining the minimum value of $64+4=\boxed{68}$.

Problem 10. Find the number of pairs of positive integers (m,n) such that $n<m\leq 100$ and the polynomial x^m+x^n+1 has a root on the unit circle.

Proposed by Ritvik Teegavarapu

Solution: $\boxed{1260}$.

Suppose $\gcd(m,n)=1$. Then, for any integer $d\geq 1$, the polynomial $f(x)=x^m+x^n+1$ has a root on the unit circle if and only if the polynomial $f(x^d)=x^{md}+x^{nd}+1$ does. To this end, we first consider the case $\gcd(m,n)=1$.

Let m,n be relatively prime positive integers such that the polynomial x^m+x^n+1 has a root r of magnitude 1. Thus, $|r^m|=|r^n|=1$, so the equation $r^m+r^n+1=0$ says that the sum of the three corresponding unit vectors in $\mathbb{R}^2$ is 0. These three vectors must necessarily form an equilateral triangle. Thus, r^m and r^n must take on $\exp(2\pi i/3)$ and $\exp(4\pi i/3)$ in some order.

By properties of complex argument, there exist integers a,b such that $m\cdot\arg(r),n\cdot\arg(r)$ equals $2\pi/3+2\pi a,4\pi/3+2\pi b$ in some order. Suppose m,n correspond to $2\pi/3+2\pi a,4\pi/3+2\pi b$, respectively. Then,

$$\frac{m}{n}=\frac{2\pi/3+2\pi a}{4\pi/3+2\pi b}=\frac{1+3a}{2+3b}\implies 3(mb-na)=n-2m.$$

Thus, there exist such integers a,b if and only if $3\mid n-2m$ (cf. Bezout's lemma). In the other case m,n correspond to $4\pi/3+2\pi b,2\pi/3+2\pi a$, respectively, the same argument shows that there exist such integers a,b if and only if $3\mid m-2n$. In either case, the integers a,b exist if and only if m,n are congruent to $1 \pmod 3$ or $2 \pmod 3$ in some order.

Now, suppose $100\geq m>n$ are positive integers, not necessarily coprime. The pair (m,n) satisfies the conditions of the problem if and only if the pair $\left(\frac{m}{\gcd(m,n)},\frac{n}{\gcd(m,n)}\right)$ satisfies the $\pmod 3$ conditions listed above. If m,n are not both divisible by 3, then dividing by $\gcd(m,n)$ certainly preserves the $\pmod 3$ conditions. Thus, we need only do casework on the possible 3-adic valuations $v_3\in\{0,1,2,3\}$ of $\gcd(m,n)$ (that is, v_3 is the exponent of 3 in the prime factorization of $\gcd(m,n)$).

There are 34 choices of an integer k such that $0\leq 3k+1\leq 100$, and 33 choices of an integer l such that $0\leq 3l+2\leq 100$. Then, these $34\cdot 33$ pairs (k,l) are in bijection with valid pairs (m,n) corresponding to $v_3=0$. By repeating this counting argument, we see that $v_3=1,2,3$ give rise to $11\cdot 11,4\cdot 4,1\cdot 1$ pairs (m,n), respectively. The answer is $34\cdot 33+11\cdot 11+4\cdot 4+1\cdot 1=\boxed{1260}$.

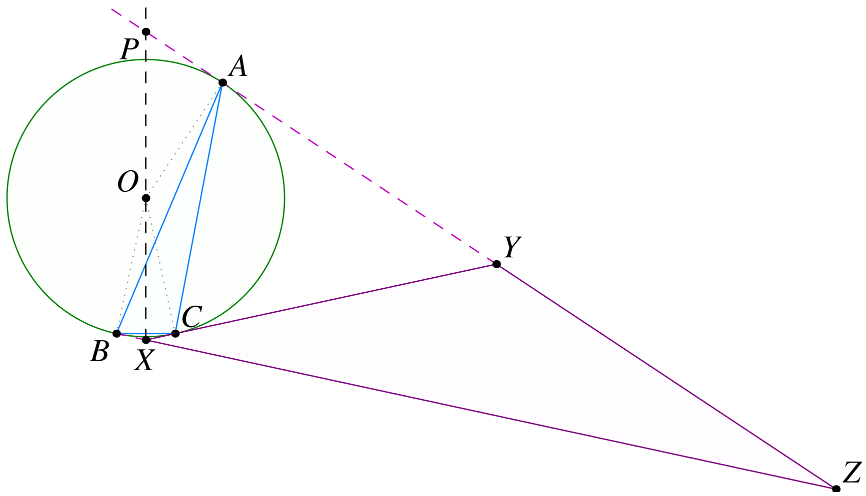
Problem 11. Let ABC be a triangle and let ω be the circle passing through A,B,C with center O. Lines l_A,l_B,l_C are drawn tangent to ω at A,B,C respectively. The intersections of these lines form a triangle XYZ where X is the intersection of l_B and l_C, Y is the intersection of l_C and l_A, and Z is the intersection of l_A and l_B. Let P be the intersection of lines $\overline{OX}$ and $\overline{YZ}$. Given $\angle ACB=\frac{3}{2}\angle ABC$ and $\frac{AC}{AB}=\frac{15}{16}$, find $\frac{ZP}{YP}$.

Proposed by August Chen

[1] Of course, for the latter three values $q=3^2,5,11$, one may also appeal to the primitive root theorem

Solution: $\boxed{\dfrac{63}{32}}$.

First, we claim that $\angle C$ in $\triangle ABC$ is obtuse.



Let $x = \frac{1}{2}\angle ABC \in (0, 90^\circ)$. We have

$$\frac{15}{16} = \frac{AC}{AB} = \frac{\sin 2x}{\sin 3x} = \frac{2\sin x\cos x}{3\sin x - 4\sin^3 x} = \frac{2\cos x}{3 - 4\sin^2 x} = \frac{2\cos x}{4\cos^2 x - 1} \implies 60\cos^2 x - 15 = 32\cos x.$$

The resulting quadratic has roots $\frac{5}{6}$ and $\frac{-3}{10}$, so $\cos x = \frac{5}{6}$. Now we can compute

$$\begin{aligned} \cos B &= \cos 2x = 2\cos^2 x - 1 = \frac{25}{18} - 1 = \frac{7}{18} \\ \cos C &= \cos 3x = 4\cos^3 x - 3\cos x = \frac{125}{54} - \frac{5}{2} = -\frac{5}{27}. \end{aligned}$$

Thus, $\angle C$ is obtuse. Note that $\overline{OX}$ is the (internal) angle bisector of $\angle BXC$, i.e. the external angle bisector of $\angle ZXY$, so by the angle bisector theorem we just want to find $\frac{XZ}{XY}$. Note $\angle XYZ = 180^\circ - \angle AYC$, $\angle AYC = 180^\circ - \angle COA = 180^\circ - 2\angle B$, so we have $\angle XYZ = 2\angle B$. On the other hand, $\angle YZX = 180^\circ - \angle AOB$, $\angle AOB = 2(180^\circ - \angle ACB)$, so we have $\angle YZX = 2\angle ACB - 180^\circ$. Therefore, we have

$$\frac{XZ}{XY} = \frac{\sin(\angle XYZ)}{\sin(\angle YZX)} = -\frac{\sin 2B}{\sin 2C} = -\frac{\sin B}{\sin C}\cdot\frac{\cos B}{\cos C} = \frac{15}{16}\cdot\frac{7}{18}\cdot\frac{27}{5} = \boxed{\frac{63}{32}}.$$

Problem 12. Compute the remainder when

$$\sum_{1\le a,k\le 2021} a^k$$

is divided by 2022 (in the above summation a, k are integers).

Proposed by Jeck Lim

Solution: $\boxed{1649}$.

First recall the following lemma:

Lemma 1: Let p be a prime, k a positive integer. Then, $\sum_{a=0}^{p-1} a^k$ is congruent to $0 \pmod p$ if $(p-1) \nmid k$; otherwise, $\sum_{a=0}^{p-1} a^k$ is congruent to $-1 \pmod p$.
Proof 1: The case $(p-1) \mid k$ follows from Fermat's theorem, so assume otherwise. Let q be a primitive root of p, so that $\{1,2,\dots,p-1\} = \{1,q,\dots,q^{p-2}\}$ modulo p. Then,

$$\sum_{a=0}^{p-1} a^k = \sum_{j=0}^{p-2} q^{jk} = \frac{q^{(p-1)k}-1}{q^k-1}$$

and since $p \nmid q^k - 1$, we see by Fermat's theorem that the RHS is indeed a multiple of p. $\square$
Throughout what follows let a,k be positive integers not greater than 2021. Write $S_k := \sum_{a=1}^{2021} a^k$; we want to find $S := \sum_{1\le a,k\le 2021} a^k = \sum_{k=1}^{2021} S_k$ modulo 2022. Since $2022 = 2\cdot 3\cdot 337$, it is enough to consider S modulo $p = 2,3,337$ by the CRT. For $p=2$, notice that for $1 \le a,k \le 2021$, a^k is odd if and only if a is odd, and this occurs for $1011\cdot 2021$ such pairs (a,k) (hence an odd number of such pairs), so S is odd.
For $p=3$, we have $p-1 \mid k$ if and only if k is even. By applying Fermat's theorem, notice S_k is $674\cdot\sum_{a=0}^{2} a^k$ modulo 3. By the above lemma, in the case k is odd, S_k is congruent to $0 \pmod 3$; in the case k is even, S_k is congruent to $-1\cdot 674 \equiv 1 \pmod 3$. There are 1010 such even k, so $S \equiv 1010 \equiv 2 \pmod 3$.
Finally, for $p = 337$, we have $p-1 \mid k$ if and only if k is a multiple of 336. Again by Fermat's theorem, notice S_k is $6\cdot\sum_{a=0}^{336} a^k$ modulo 337. In case k is a multiple of 336, this is congruent to $-1\cdot 6 \equiv -6 \pmod{337}$; otherwise, this is congruent to $0 \pmod{337}$. There are 6 such k that are multiples of 336, so $S \equiv -6\cdot 6 \equiv -36 \pmod{337}$.
Putting everything together, we compute $S \equiv \boxed{1649} \pmod{2022}$.

Problem 13. Consider a 7×2 grid of squares, each of which is equally likely to be colored either red or blue. Madeline would like to visit every square on the grid exactly once, starting on one of the top two squares and ending on one of the bottom two squares. She can move between two squares if they are adjacent or diagonally adjacent. What is the probability that Madeline may visit the squares of the grid in this way such that the sequence of colors she visits is *alternating* (i.e., red, blue, red, ... or blue, red, blue, ...)?

Proposed by Mathus Leungpathomaram

Solution: $\boxed{\dfrac{43}{1024}}$.
Lemma 1: There is an red-starting, blue-ending alternating path on an $n\times 2$ grid if and only if it is possible to divide the grid into 1×2 and 2×2 rectangles such that each 1×2 rectangle is colored RB or BR and each 2×2 rectangle is colored top row RR and bottom row BB.
Proof 1: First suppose the grid may be divided into 1×2 and 2×2 rectangles in this way. We can construct a red-starting alternating path recursively. Assume every square of the first $k \ge 0$ rows of the grid have been visited by the recursively-constructed alternating path. So far, we have visited an equal number of red and blue squares, so the next row must be colored RB or BR (corresponding to the 1×2 case) or RR (corresponding to the 2×2 case). In the former case, we may extend the alternating path to all squares in $k+1$ rows by appending the red square and then the blue square of row $k+1$. In the latter case, row $k+2$ is necessarily BB, so we may extend the alternating path to all squares in $k+2$ rows by appending a red square of row $k+1$, a blue square of row $k+2$, the remaining red square of row $k+1$, and finally the remaining blue square of row $k+2$. This completes the recursive construction.
Conversely, suppose P is a red-starting alternating path on the $n\times 2$ grid. Again we can apply recursion, this time to choose the desired rectangles. Suppose the first $k \ge 0$ rows of the grid have been partitioned into

rectangles as described above, and that the first $2k$ squares visited by P are precisely those of the first k rows. The next square P visits (square $2k+1$) is necessarily red, in row $k+1$. Then, for square $2k+2$, P may either visit the remaining square in row $k+1$, which is hence blue, or visit a square in row $k+2$, which is hence blue. In the latter case, the squares $2k+3$ and $2k+4$ of P must necessarily be the remaining square in row $k+1$ and then the remaining square in row $k+2$, which are red and blue, respectively. In particular, the former case shows row $k+1$ is a 1×2 rectangle colored RB or BR, while the latter case shows rows $k+1, k+2$ is a 2×2 rectangle colored top row RR and bottom row BB. The recursion is complete. $\square$

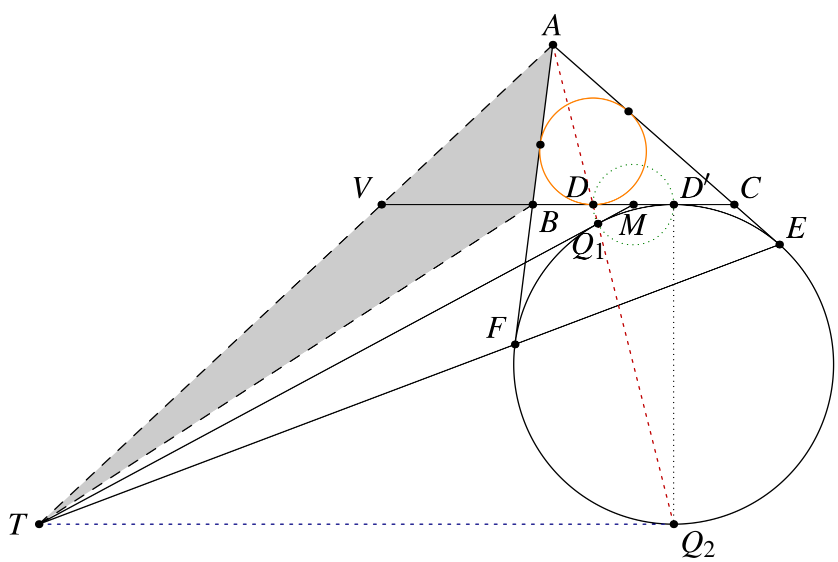
Let $f(n)$ be the number of red-starting alternating paths on an $n\times 2$ grid. We have $f(0)=1, f(1)=2$, and the above lemma implies we have the recurrence $f(n)=f(n-2)+2f(n-1)$. We compute $f(7)=408$. Since alternating paths may start on either color, we double this number to 816. However, we are double counting the cases where we only fill in 1 by 2 rectangles, of which there are $2^7=128$ of, for an actual total of 688 colorings with an alternating path. Since there are 2^{14} ways to color a 7×2 grid, we obtain an answer of $\frac{688}{2^{14}}=\boxed{\frac{43}{1024}}$.

Problem 14. Let ABC be a triangle with $AB=8$, $BC=10$, and $CA=12$. Denote by Ω_A the A-excircle of ABC, and suppose that Ω_A is tangent to $\overline{AB}$ and $\overline{AC}$ at F and E, respectively. Line $l\neq\overline{BC}$ is tangent to Ω_A and passes through the midpoint of $\overline{BC}$. Let T be the intersection of $\overline{EF}$ and l. Compute the area of triangle ATB.

Proposed by Brian Yang

Solution: $\boxed{\frac{135\sqrt{7}}{4}}$.

Consider a line $m\neq\overline{BC}$ parallel to BC and tangent to Ω_A at the point Q_2.



We claim $T\in m$. Indeed, let D be the A-intouch point and D' be the A-extouch point of $\triangle ABC$. Let M be the midpoint of BC, and suppose $\{Q_1, Q_2'\}=\overline{AD}\cap\Omega_A$ such that Q_1 is in between A and Q_2'. Notice $Q_2'=Q_2$, because the homothety $\mathscr{H}$ taking Ω_A to the incircle of $\triangle ABC$ maps m to $\overline{BC}$, hence maps Q_2 to D. Now, note $\overline{Q_2D'}$ is the diameter of Ω_A orthogonal to $\overline{BC}$, so $\overline{Q_1Q_2}\perp\overline{Q_1D'}$. However $MD=MD'$, so M is the center of (DQ_1D'), implying that $MQ_1=MD'$ and so $Q_1=l\cap\Omega_A$. Now, the polar of A is $\overline{EF}$ and the polar of Q_1 is l, so the polar of T is $\overline{AQ_1} = \overline{AQ_2}$ (all polars w.r.t. Ω_A). Thus, T lies on the polar of Q_2, which is precisely m, showing the desired claim.

Denote by V the image of T under the homothety $\mathscr{H}$. Since $\mathscr{H}$ maps Q_2 to D (as previously discussed) and maps E and F to the B- and C-intouch points in $\triangle ABC$, respectively, we deduce V is the harmonic conjugate of D with respect to B and C.

We are ready to do computations. Let a,b,c be the side lengths of $\triangle ABC$ and s its semiperimeter. Since $DB = s-b = 3, DC = s-c = 7$, we have $VB = \frac{15}{2}$ by the cross ratio $(V,D;B,C) = -1$. Computing $[ABC] = 15\sqrt{7}$ (by Heron's formula, for instance), the A-altitude of $\triangle ABC$ has length $2 \cdot \frac{[ABC]}{a} = 3\sqrt{7}$. Moreover, the A-exradius is $\frac{s}{s-a} = 3$ times the inradius (of $\triangle ABC$), so $TA = 3VA$. Thus,

$$[ATB] = 3[AVB] = 3\left(\frac{1}{2}\cdot 3\sqrt{7}\cdot\frac{15}{2}\right) = \boxed{\frac{135\sqrt{7}}{4}}.$$

Problem 15. For any positive integer n, let D_n be the set of ordered pairs of positive integers (m,d) such that d divides n and $\gcd(m,n) = 1,\ 1 \le m \le n$. For any positive integers a,b, let $r(a,b)$ be the non-negative remainder when a is divided by b. Denote by S_n the sum

$$S_n = \sum_{(m,d)\in D_n} r(m,d).$$

Determine the value of S_{396}.

Proposed by Brian Yang

Solution: $\boxed{65460}$.

Let ϕ be Euler's function, P_n be the set of all positive integers $1 \le m < n$ such that $\gcd(m,n) = 1$, and $T_{n,d} = \sum_{m\in P_n} r(m,d)$. Note $T_{n,1} = 0$ simply because division by 1 yields no remainder. The critical claim is that for $1 < d \mid n$, we have

$$T_{n,d} = \sum_{m\in P_n} r(m,d) = \frac{d\phi(n)}{2}.$$

We begin with the following lemma:

Lemma 1: Let $n > 1$ be a positive integer, $1 < d$ a divisor of n and $1 \le m < n$ a positive integer satisfying $\gcd(m,n) = 1$. The number of positive integers $1 \le m' < n$ such that $\gcd(m',n) = 1$ and $m' \equiv m \pmod{d}$ equals $\frac{\phi(n)}{\phi(d)}$.

Proof 1: Let n_1 be the largest divisor of n such that n_1, d have the same prime divisors, and let $n_2 = \frac{n}{n_1}$. Then, notice that n_1 and n_2 are coprime. Then, for any pair of integers $0 \le a_1 < n_1, 0 \le a_2 < n_2$, there exists (by the CRT) a unique integer $1 \le m' \le n$ such that $m' \equiv a_i \pmod{n_i}$, $i = 1,2$. Thus, a pair (a_1,a_2) gives rise to an m' satisfying the condition of the lemma if and only if $\gcd(a_i,n_i) = 1$, $i = 1,2$ (this is equivalent to $\gcd(m',n_i) = 1$, $i = 1,2$, or $\gcd(m',n) = 1$), and $a_1 \equiv m \pmod{d}$. There are $\frac{n_1}{d}$ integers $0 \le a_1 < n_1$ giving rise to $a_1 \equiv m \pmod{d}$. For each such choice of a_1, we have $\gcd(a_1,n_1) = \gcd(a_1,d) = \gcd(m,d) = 1$ where the first equality is justified as n_1, d have the same prime divisors. This gives us $\frac{n_1}{d}$ possible choices of a_1. On the other hand, there are $\phi(n_2)$ choices of $0 \le a_2 < 1$ such that $\gcd(a_2,n_2) = 1$. It follows that the total number of pairs (a_1,a_2) (and hence the total number of integers $1 \le m' < n$) satisfying the condition of the lemma) is $\frac{n_1\phi(n_2)}{d}$. However, note $\frac{\phi(n_1)}{\phi(d)} = \frac{n_1}{d}$ as n_1, d have the same prime divisors, and $\phi(n_1)\phi(n_2) = \phi(n)$ by multiplicativity of ϕ, so the total number of integers $1 \le m' < n$ satisfying the conditions equals $\frac{\phi(n_1)\phi(n_2)}{\phi(d)} = \frac{\phi(n)}{\phi(d)}$, as requested. $\square$

Thus, for each integer $a \in P_d$, there exist $\frac{\phi(n)}{\phi(d)}$ integers $1 \le m < n$ such that $m \equiv a \pmod{d}$. That is, each integer $a \in P_d$ appears $\frac{\phi(n)}{\phi(d)}$ times in the sum $T_{n,d}$. Now we consider the set P_d. Start with the case $d > 2$. For $a \in P_d$, note that if $\gcd(a,d) = 1$, then $d - a \neq a$ and $\gcd(d-a,d) = 1$, i.e. the set P_d may be partitioned into (unordered) pairs $(a, d-a)$ such that the sum of each pair equals d. Thus, the sum of all integers in P_d equals $\frac{d\phi(d)}{2}$. This formula also holds in case $d = 2$. Putting everything together, we deduce that

$$T_{n,d} = \sum_{m \in P_n} r(m,d) = \frac{\phi(n)}{\phi(d)} \cdot \frac{d\phi(d)}{2} = \frac{d\phi(n)}{2}$$

for $1 < d \mid n$, as claimed. To finish, denote by $\sigma(n)$ the sum of the positive divisors of n. Then, we have

$$S_n = \sum_{1 \le d \mid n} T_{n,d} = -\frac{\phi(n)}{2} + \sum_{1 \le d \mid n} \frac{d\phi(n)}{2} = \frac{\phi(n)}{2}(\sigma(n) - 1).$$

Now, let $n = 396 = 2^2 \cdot 3^2 \cdot 11$. Since $\phi(396) = 120$ and $\sigma(396) = (1+2+2^2)(1+3+3^2)(1+11) = 1092$, the answer is $S_{396} = \boxed{65460}$.