Caltech Harvey Mudd Mathematics Competition

Mixer Round **November 13, 2010**

In this round, problems will depend on the answers to other problems. A bolded letter is used to denote a quantity whose value is determined by another problem's answer.

Part I

1. Let **F** be the answer to problem number 6. You want to tile a nondegenerate square with side length **F** with 1×2 rectangles and 1×1 squares. The rectangles can be oriented in either direction. How many ways can you do this?

2. Let **A** be the answer to problem number 1. Triangle ABC has a right angle at B and the length of AC is **A**. Let D be the midpoint of AB, and let P be a point inside triangle ABC such that $PA = PC = \frac{7\sqrt{5}}{4}$ and $PD = \frac{7}{4}$. The length of AB^2 is expressible as m/n for m, n relatively prime positive integers. Find m.

3. Let **B** be the answer to problem number 2. Let S be the set of positive integers less than or equal to **B**. What is the maximum size of a subset of S whose elements are pairwise relatively prime?

4. Let **C** be the answer to problem number 3. You have 9 shirts and 9 pairs of pants. Each is either red or blue, you have more red shirts than blue shirts, and you have same number of red shirts as blue pants. Given that you have **C** ways of wearing a shirt and pants whose colors match, find out how many red shirts you own.

5. Let **D** be the answer to problem number 4. You have two odd positive integers a, b. It turns out that $\text{lcm}(a, b) + a = \gcd(a, b) + b = \mathbf{D}$. Find ab.

6. Let **E** be the answer to problem number 5. A function f defined on integers satisfies $f(y)+f(12-y) = 10$ and $f(y) + f(8 - y) = 4$ for all integers y. Given that $f(\mathbf{E}) = 0$, compute $f(4)$.

Part II

7. Let **L** be the answer to problem number 12. You want to tile a nondegenerate square with side length **L** with 1×2 rectangles and 7×7 squares. The rectangles can be oriented in either direction. How many ways can you do this?

8. Let **G** be the answer to problem number 7. Triangle ABC has a right angle at B and the length of AC is **G**. Let D be the midpoint of AB, and let P be a point inside triangle ABC such that $PA = PC = \frac{1}{2}$ and $PD = \frac{1}{2010}$. The length of AB^2 is expressible as m/n for m, n relatively prime positive integers. Find m.

9. Let **H** be the answer to problem number 8. Let S be the set of positive integers less than or equal to **H**. What is the maximum size of a subset of S whose elements are pairwise relatively prime?

10. Let **I** be the answer to problem number 9. You have 391 shirts and 391 pairs of pants. Each is either red or blue, you have more red shirts than blue shirts, and you have same number of red shirts as red pants. Given that you have **I** ways of wearing a shirt and pants whose colors match, find out how many red shirts you own.

11. Let **J** be the answer to problem number 10. You have two odd positive integers a, b. It turns out that $\text{lcm}(a, b) + 2a = 2\gcd(a, b) + b = \mathbf{J}$. Find ab.

12. Let **K** be the answer to problem number 11. A function f defined on integers satisfies $f(y)+f(7-y) = 8$ and $f(y) + f(5 - y) = 4$ for all integers y. Given that $f(\mathbf{K}) = 453$, compute $f(2)$.