Tiebreaker Solutions

2018

1. A large pond contains infinitely many lily pads labelled $1, 2, 3, \ldots$, placed in a line, where for each k, lily pad $k+1$ is one unit to the right of lily pad k. A frog starts at lily pad 100. Each minute, if the frog is at lily pad n, it hops to lily pad $n+1$ with probability $\frac{n-1}{n}$, and hops all the way back to lily pad 1 with probability $\frac{1}{n}$. Let N be the position of the frog after 1000 minutes. What is the expected value of N?

   Solution: Let X_k be the position of the frog after k minutes have passed. For any n, the expected value of X_{k+1}, given that $X_k = n$, is $\frac{n-1}{n}(n+1) + \frac{1}{n}(1) = n$. It follows that the expected value of X_{k+1} is equal to the expected value of X_k, meaning this value does not change from minute to minute. Since the expected value of X_0 is 100, the expected value of $N = X_{1000}$ is $\boxed{100}$ as well.

2. A cat is tied to one corner of the base of a tower. The base forms an equilateral triangle of side length 4 m, and the cat is tied with a leash of length 8 m. Let A be the area of the region accessible to the cat. If we write $A = \frac{m}{n}\pi - k\sqrt{\ell}$, where m, n, k, ℓ are positive integers such that m and n are relatively prime, and ℓ is squarefree, what is the value of $m + n + k + \ell$?

   Note: the original wording mistakenly had $A = \frac{m}{n}\pi + k\sqrt{\ell}$.

   Solution: Say the tower's base is triangle BCD, with the cat tied at point B. The cat can only move within the circle of radius 8 m around point B, and outside the sector of this circle spanned by angle $\angle CBD$ the cat can move anywhere in the circle. However, in this sector, the cat's motion is constrained further: it can only move within 4 m of either C or D, i.e. it must be contained within one of the two circles centered at C and D with radius 4 m. If the cat somehow moved outside these circles within the sector, then tightening the leash would yield a path consisting of two segments, one from B to either C or D, and the second from the latter point to the cat. The first segment would have length 4 m, while the second would have length exceeding 4 m, for a total length exceeding 8 m, which is impossible. Constructing this figure formed by three circles, we see the area is $\frac{5}{6}(64\pi) + 2(\frac{1}{3})(16\pi) - 2(\frac{1}{6}(16\pi) - \frac{1}{2}(2)(2\sqrt{3})) = \frac{176}{3}\pi - 4\sqrt{3}$. This gives $m + n + k + \ell = \boxed{186}$.

3. Compute
$$\sum_{n=1}^{\infty} \left(\frac{1}{n^2 + 3n} - \frac{1}{n^2 + 3n + 2} \right)$$

   Solution: We have $\frac{1}{n^2+3n} = \frac{1}{3}(\frac{1}{n} - \frac{1}{n+3})$ and $\frac{1}{n^2+3n+2} = \frac{1}{n+1} - \frac{1}{n+2}$, so the sum telescopes to $\frac{1}{3}(\frac{1}{1} + \frac{1}{2} + \frac{1}{3}) - \frac{1}{2} = \boxed{1/9}$.

4. Find the sum of the real roots of $f(x) = x^4 + 9x^3 + 18x^2 + 18x + 4$.

   Solution: Note $f(x)$ factors as $(x^2 + 7x + 2)(x^2 + 2x + 2)$, and only $x^2 + 7x + 2$ has real roots, which sum to $\boxed{-7}$.

5. Let a, b, c, d, e be the roots of $p(x) = 2x^5 - 3x^3 + 2x - 7$. Find the value of

$$(a^3 - 1)(b^3 - 1)(c^3 - 1)(d^3 - 1)(e^3 - 1)$$

Solution: We have

$$\prod(a^3 - 1) = \prod(a - 1)(a - \omega)(a - \omega^2) = -\frac{1}{8}p(1)p(\omega)p(\omega^2)$$

where $\omega = e^{2\pi i/3}$. Evaluating these, using the facts that $\omega^2 + \omega + 1 = 0$, and $\omega^3 = 1$, we have $p(1) = -6$, $p(\omega) = 2\omega^2 - 3 + 2\omega - 7 = -12$, $p(\omega^2) = 2\omega - 3 + 2\omega^2 - 7 = -12$, so the desired product is $-\frac{1}{8}(-6)(-12)(-12) = \boxed{108}$.