Team Round 2022-2023

Problem 1. A wall contains three switches A, B, C, each of which powers a light when flipped on. Every 20 seconds, switch A is turned on and then immediately turned off again. The same occurs for switch B every 21 seconds and switch C every 22 seconds. At time $t = 0$, all three switches are simultaneously on.

Let $t = T > 0$ be the earliest time that all three switches are once again simultaneously on. Compute the number of times $t > 0$ before T when at least two switches are simultaneously on.

Problem 2. Select a number X from the set of all 3-digit natural numbers uniformly at random. Let $A \in [0,1]$ be the probability that X is divisible by 11, given that it is palindromic. Let $B \in [0,1]$ be the probability that X is palindromic, given that it is divisible by 11. Compute $B - A$.

Recall that a 3-digit number is a palindrome if it reads the same left to right as right to left. For instance, 484 *is a palindrome, but* 603 *is not a palindrome.*

Problem 3. Let $a_1, a_2, \ldots$ be a strictly increasing sequence of positive real numbers such that $a_1 = 1, a_2 = 4$, and that for every positive integer k, the subsequence $a_{4k-3}, a_{4k-2}, a_{4k-1}, a_{4k}$ is geometric and the subsequence $a_{4k-1}, a_{4k}, a_{4k+1}, a_{4k+2}$ is arithmetic. For each positive integer k, let r_k be the common ratio of the geometric sequence $a_{4k-3}, a_{4k-2}, a_{4k-1}, a_{4k}$. Compute

$$\sum_{k=1}^{\infty} (r_k - 1)(r_{k+1} - 1).$$

Problem 4. Gus is an inhabitant on an 11 by 11 grid of squares. He can walk from one square to an adjacent square (vertically or horizontally) in 1 unit of time. There are also two vents on the grid, one at the top left and one at the bottom right. If Gus is at one vent, he can teleport to the other vent in 0.5 units of time. Let an ordered pair of squares (a, b) on the grid be *sus* if the fastest path from a to b requires Gus to teleport between vents. Walking on top of a vent does not count as teleporting between vents.

What is the total number of ordered pairs of squares that are *sus*? Note that the pairs (a_1, b_1) and (a_2, b_2) are considered distinct if and only if $a_1 \neq a_2$ or $b_1 \neq b_2$.

Problem 5. Let ABC be a triangle with $AB = 6, AC = 8, BC = 7$. Let H be the orthocenter of ABC. Let $D \neq H$ be a point on $\overline{AH}$ such that $\angle HBD = \frac{3}{2}\angle CAB + \frac{1}{2}\angle ABC - \frac{1}{2}\angle BCA$. Find DH.

Problem 6. Let A be a set of 8 elements, and $\mathscr{B} := (B_1, \ldots, B_7)$ be an ordered 7-tuple of subsets of A. Let N be the number of such 7-tuples $\mathscr{B}$ such that there exists a unique 4-element subset $I \subseteq \{1, 2, \ldots, 7\}$ for which the intersection $\bigcap_{i \in I} B_i$ is nonempty. Find the remainder when N is divided by 67.

Problem 7. Let $\mathbb{N}_0$ be the set of all non-negative integers. Let $f : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ be a function such that for all non-negative integers a, b:

$$f(a,b) = f(b,a),$$
$$f(a,0) = 0,$$
$$f(a+b,b) = f(a,b) + b.$$

Compute

$$\sum_{i=0}^{30} \sum_{j=0}^{2^i - 1} f(2^i, j).$$

Problem 8. Suppose $a_3x^3 - x^2 + a_1x - 7 = 0$ is a cubic polynomial in x whose roots α, β, γ are positive real numbers satisfying

$$\frac{225\alpha^2}{\alpha^2+7} = \frac{144\beta^2}{\beta^2+7} = \frac{100\gamma^2}{\gamma^2+7}.$$

Find a_1.

Problem 9. Let $ABCD$ be a convex, non-cyclic quadrilateral with E the intersection of its diagonals. Given $\angle ABD + \angle DAC = \angle CBD + \angle DCA$, $AB = 10$, $BC = 15$, $AE = 7$, and $EC = 13$, find BD.

Problem 10. Suppose that $\xi \neq 1$ is a root of the polynomial $f(x) = x^{167} - 1$. Compute

$$\left| \sum_{0<a<b<167} \xi^{a^2+b^2} \right|.$$

In the above summation a, b are integers.