Proof Round 2024-2025

Problem 1. [5] Does there exist a prime p and distinct positive integers y_1, y_2, y_3 with $y_i < \frac{p}{3}$ for each $1 \le i \le 3$ and $y_1^3 \equiv y_2^3 \equiv y_3^3 \mod p$?

Problem 2. [5] The n vertices of a regular n-gon are drawn, and each of the $\binom{n}{2}$ edges between them is colored such that no two edges of the same color intersect at a point which is not a vertex of the n-gon. What is the minimum number of different colors that can be used to color all the edges?

Problem 3. [10] Let p be a prime and S be the set of positive integers m such that the only factors of m which are congruent to $1 \pmod{p}$ are 1 and m. Suppose we have some $m \in S$ such that m has $p-1$ distinct prime factors. Show that $m = q_1 \cdots q_{p-1}$ for distinct primes $q_1, \cdots, q_{p-1}$ such that $q_1 \equiv \cdots \equiv q_{p-1} \mod p$.

Problem 4. [10] In triangle ABC, points E and F are chosen on sides AC and AB respectively such that $AE = AF$. Point K is on segment EF such that AK, BE, and CF intersect at a point, and points B' and C' are on lines AC and AB such that $BB' \parallel CK$ and $CC' \parallel BK$. Points U and V are on the line through A parallel to EF such that $UC = UC'$ and $VB = VB'$. Prove that $AU = AV$.

Problem 5. [10] Let $\mathbb{Q}$ denote the set of rational numbers. Suppose $f : \mathbb{Q} \to \mathbb{Q}$ is a function that satisfies $0 < |f(x) - f(y)| \leqslant |x - y|$ for all $x \neq y \in \mathbb{Q}$. Must f be monotonic?

A function f is increasing if $f(x) < f(y)$ whenever $x < y$. A function f is monotonic if either f or $-f$ is increasing.

Problem 6. [15] Let a, b, c, and d be positive integers such that

- $ab = c^2 + d$
- $2c \mid a - b$
- $cd < a - b$

Prove that d is a perfect square.

Problem 7. [20] Puddles the Frog has a row of $n \geq 2$ lilypads, labeled 1 through n, with 1 unit between consecutive lilypads. He is at lilypad 1, and wishes to visit lilypads 2 through n once each, before ending up back at lilypad 1. However, he may jump no more than 3 units at a time. Let $f(n)$ be the maximum total distance of Puddles's trip, starting from lilypad 1 and ending back at lilypad 1. Determine, with proof, the value of

$$\lim_{n \to \infty} \frac{f(n)}{n}.$$