CHMMC 2021-2022

Rules

1. You have 120 minutes to complete the test.
2. You may collaborate with your other teammates on this round. Otherwise, **no other collaboration, computers, calculators, or other outside aid is permitted** besides what is listed below.
3. You are, however, permitted to use ruler and compass but **not** a protractor.
4. You can message an organizer **individually** in Slack to ask *only clarifying questions*. You may not receive a response if the question may help you solve the problems.
5. This round will count towards 40% of the group round score, the proof round being worth the other 60%. This group round score in turn is half the team score.
6. The top 10 teams will be recognized, where the procedure the break ties has been detailed on the Chmmc website.
7. All the answers to each problem are integers.

Team Round

Problem 1. Let ABC be a right triangle with hypotenuse $\overline{AC}$ and circumcenter O. Point E lies on $\overline{AB}$ such that $AE = 9, EB = 3$; point F lies on $\overline{BC}$ such that $BF = 6, FC = 2$. Now suppose W, X, Y, and Z are the midpoints of $\overline{EB}$, $\overline{BF}$, $\overline{FO}$, and $\overline{OE}$, respectively. Compute the area of quadrilateral $WXYZ$.

Problem 2. A prefrosh is participating in Caltech's "Rotation." They must rank Caltech's 8 houses, which are Avery, Page, Lloyd, Venerable, Ricketts, Blacker, Dabney, and Fleming, each a distinct integer rating from 1 to 8 inclusive. The conditions are that the rating x they give to Fleming is at most the average rating y given to Ricketts, Blacker, and Dabney, which is in turn at most the average rating z given to Avery, Page, Lloyd, and Venerable. Moreover x, y, z are all integers. How many such rankings can the prefrosh provide?

Problem 3. Suppose a, b, c are complex numbers with $a + b + c = 0, a^2 + b^2 + c^2 = 0$, and $|a|, |b|, |c| \leq 5$. Suppose further at least one of a, b, c have real and imaginary parts that are both integers. Find the number of possibilities for such ordered triples (a, b, c).

Problem 4. How many ordered triples (a, b, c) of integers $1 \leq a, b, c \leq 31$ are there such that the remainder of $ab + bc + ca$ divided by 31 equals 8?

Problem 5. How many cubics in the form $x^3 - ax^2 + (a + d)x - (a + 2d)$ for integers a, d have roots that are all non-negative integers?

Problem 6. There is a unique degree-10 monic polynomial with integer coefficients $f(x)$ such that

$$f\left(\sum_{j=0}^{9} \sqrt[10]{2021^j}\right) = 0.$$

Find the remainder when $f(1)$ is divided by 1000.

Problem 7. Let ABC be a triangle with $AB = 5$, $BC = 6$, and $CA = 7$. Denote Γ the incircle of ABC; let I be the center of Γ. The circumcircle of BIC intersects Γ at X_1 and X_2. The circumcircle of CIA intersects Γ at Y_1 and Y_2. The circumcircle of AIB intersects Γ at Z_1 and Z_2. The area of the triangle determined by $\overline{X_1X_2}$, $\overline{Y_1Y_2}$, and $\overline{Z_1Z_2}$ equals $\frac{m\sqrt{p}}{n}$ for positive integers m, n, and p, where m and n are relatively prime and p is squarefree. Compute $m + n + p$.

Problem 8. Depei is imprisoned by an evil wizard and is coerced to play the following game. Every turn, Depei flips a fair coin. Then, the following events occur in this order:

- The wizard computes the difference between the total number of heads and the total number of tails Depei has flipped. If that number is greater than or equal to 4 or less than or equal to -3, then Depei is vaporized by the wizard.
- The wizard determines if Depei has flipped at least 10 heads or at least 10 tails. If so, then the wizard releases Depei from the prison.

The probability that Depei is released by the evil wizard equals $\frac{m}{2^k}$, where m, k are positive integers. Compute $m + k$.

Problem 9. Find the largest prime divisor of

$$\sum_{n=3}^{30} \binom{\binom{n}{3}}{2}.$$

Problem 10. In triangle ABC, let O be the circumcenter. The incircle of ABC is tangent to $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at points D, E, and F, respectively. Let G be the centroid of triangle DEF. Suppose the inradius and circumradius of ABC is 3 and 8, respectively. Over all such triangles ABC, pick one that maximizes the area of triangle AGO. If we write $AG^2 = \frac{m}{n}$ for relatively prime positive integers m and n, then find m.