CMM 2026 Individual Round Solutions

Caltech Math Meet

January 24, 2026

1 Problem 1

Problem 1. The Fibonacci pirates are a motley gang of 7 pirates, Alpha, Beta, Gamma, Delta, Epsilon, Zeta, and Eta. They have a stash of $N > 0$ identical coins that they would like to divide up between themselves. They arrange themselves in a line in some order, and each pirate, when it is their turn to take their portion from the stash, does so with the following rules:

- Alpha takes all of what remains in the stash.
- Beta takes $\frac{1}{2}$ of what remains in the stash.
- Gamma takes $\frac{1}{3}$ of what remains in the stash.
- Delta takes $\frac{1}{5}$ of what remains in the stash.
- Epsilon takes $\frac{1}{8}$ of what remains in the stash.
- Zeta takes $\frac{1}{13}$ of what remains in the stash.
- Eta takes $\frac{1}{21}$ of what remains in the stash.
- Alpha, being the greediest of the pirates, is confined to go last.

Suppose the pirates arrange themselves in some order so that each pirate receives an integer amount of coins. What is the fewest possible number of coins that Epsilon receives?

Proposed by Justin Lee

Answer: $\boxed{3}$.

1.1 Solution

Consider the number of coins that Alpha receives: after every other pirate takes their share of the stash, Alpha is left with

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{7}{8} \cdot \frac{12}{13} \cdot \frac{20}{21} \cdot N = \frac{8}{39}N$$

It is therefore evident that $39 \mid N$. Observe that there exists an arrangement with $N = 39$: for instance, an ordering such as Zeta, Gamma, Epsilon, Eta, Delta, Beta, and Alpha is valid, for it yields the following sequence of number of coins left in the stash after each turn:

$$39 \to 36 \to 24 \to 21 \to 20 \to 16 \to 8 \to 0$$

In this arrangement, Epsilon receives 3 coins. To show that this is the fewest possible number of coins, observe that after Epsilon's turn, the number of coins left in the stash must be divisible by 7. If Eta went before Epsilon, then every subsequent turn (after Epsilon's) must leave a multiple of 7 number of coins in the stash. Consequently, $39 \cdot 7 \mid N$ which means that Epsilon must have left at least $\frac{8}{39}N \geq 56$ coins left, meaning Epsilon received at least 7 coins. On the other hand, if Eta went after Epsilon, then there must be at least 21 coins left in the stash after Epsilon's turn, which means Epsilon must have received at least 3 coins.

2 Problem 2

Problem 2. The arithmetic sequence $a_1, a_2, \dots$ with $a_1 \neq a_2$ satisfies that a_1, a_2, a_n and a_1, a_4, a_p are both geometric sequences. In this case, p can be represented as $cn + d$ for some integers c and d. Compute $c \cdot d$.

Proposed by Amudhan Gurumoorthy

Answer: $\boxed{-180}$.

2.1 Solution

Let a_0 be the term of the arithmetic sequence "right before" a_1. then we can write a_1, a_2, a_n as $a_0 + x, a_0 + 2x, a_0 + nx$ and a_1, a_4, a_p as $a_0 + x, a_0 + 4x, a_0 + px$, so the geometric sequences give us the equations:

$$\tfrac{a_0+nx}{a_0+2x} = \tfrac{a_0+2x}{a_0+x} \text{ and } \tfrac{a_0+px}{a_0+4x} = \tfrac{a_0+4x}{a_0+x}$$

Simplifying this expression and canceling terms results in:

$$a_0(n+1) + nx = 4a_0 + 4x \text{ and } a_0(p+1) + px = 8a_0 + 16x$$

which then simplifies to:

$$n(a_0 + x) = 3a_0 + 4x \text{ and } p(a_0 + x) = 7a_0 + 16x$$

Now, we can write n and p as $\frac{3a_0+4x}{a_0+x} = 3 + \frac{x}{a_0+x}$ and $\frac{7a_0+16x}{a_0+x} = 7 + \frac{9x}{a_0+x}$. Thus, we can see that $n = 9p - 20$, so the answer is $9 \cdot (-20) = -180$.

3 Problem 3

Problem 3. Ryan has a circular lock consisting of five equally spaced numbers, each a distinct digit from 1 to 5. Ryan does not know the correct code, so he inputs a random permutation of $1, \cdots, 5$ in the five slots. He can turn the lock by an arbitrary multiple of $72°$, which rotates the positions of the numbers he inputted (without changing the value of the correct code). What is the probability that there is some turn of the lock (possibly $0°$) for which at least two digits are in the correct positions?

Proposed by Justin Lee

Answer: $\boxed{\frac{7}{8}}$.

3.1 Solution

For each rotation of the lock, consider the number of digits that are in the correct position. If we sum this number across all five rotations, we must obtain 5, since each digit will be in the correct position exactly once. Therefore, the complement of the answer is the probability that each turn of the lock yields exactly one digit in the correct position.

The rotational symmetry of the lock allows us to assume that the digit 1 is in the correct position in the starting configuration; under this assumption, there are 24 total configurations. We now count the number of such configurations with the property noted above, namely that each rotation of the lock yields exactly one digit in the correct position.

For simplicity, assume that the correct code is 1-2-3-4-5. Now we do casework based on the number that is in the correct position after one turn of the lock, represented by a right shift in the sequence. Note that this number cannot be 1 or 2.

- If the number is 3, then Ryan's input is of the form 1-3-x-y-z. Nowx y cannot be 4 or 5, so $y = 2$, and we deduce $x = 5$ and $z = 4$.
- If the number is 4, then Ryan's input is of the form 1-x-4-y-z. Now, x cannot be 2 or 3, so $x = 5$ and we deduce $y = 3$ and $z = 2$.
- If the number is 5, then Ryan's input is of the form 1-x-y-5-z. Now, x cannot be 2 or 3, so $x = 4$ and we deduce $y = 2$ and $z = 3$.

In summary, there are 3 configurations with the above property, so the answer is $1 - \frac{3}{24} = \frac{7}{8}$.

4 Problem 4

Problem 4. For positive integers n, let $f(n)$ denote the number whose base-3 expansion coincides with the binary expansion of n. Compute the number of positive integers less than or equal to 2026 that can be written in the form $f(a+b) - f(a) - f(b)$ for some positive integers a and b.

Proposed by Justin Lee

Answer: $\boxed{127}$.

4.1 Solution

Lemma 4.1
A base-3 number can be written in the form $f(a+b) - f(a) - f(b)$ if and only if its base-3 representation has only the digits 0 and 1.

Proof. We will examine what happens when we add two binary strings and then apply f to turn everything into base 3. The only situation in which addition in base 2 (with 1's and 0's) is not the same as addition in base 3 is when we have a carry in base 2. All these carries are of the form $1_2 + 1_2 = 10_2$ (or the same thing with zeros appended onto the end, like $100_2 + 100_2 = 1000_2$), or $1_2 + 11_2 = 100_2$ (or the same thing with zeros appended on the end), or $1_2 + 111_2 = 1000_2$ (or the same thing with zeros appended on the end), etc. Notice that $1_3 + 1_3 + 1_3 = 10_3$, $1_3 + 11_3 + 11_3 = 100_3$, $1_3 + 111_3 + 111_3 = 1000_3$, etc. Thus, the amount that $f(a+b) - f(a) - f(b)$ is incremented by with each carry is a string of 1's that is contained in the place values of that carry. For example, if $a = 10001001_2$ and $b = 01111011_2$, then $f(a+b) - f(a) - f(b) = 1111011_3$, with the first string of 1's ("1111") coming from the $1_2 + 1111_2$ carrying, and the second string of a's ("11") coming from the $1_2 + 11_2$ carrying. Note that these strings of 1's never interfere, since by definition the carries $1_2 + 1_2 = 10_2$, $1_2 + 11_2 = 100_2$, etc. do not interfere with each other. Thus, all base-3 numbers of the form $f(a+b) - f(a) - f(b)$ must only consist of 1's and 0's. Furthermore, we can backward engineer values of a and b which make $f(a+b) - f(a) - f(b)$ equal to any given number made out of 0's and 1's in base 3, by assigning carries to any strings of consecutive 1's. For example, 110010111_3 is equal to $f(a+b) - f(a) - f(b)$ where $a = 110010111_2$ (just the original string, but in base 2) and $b = 010010001_2$ (all the runs of consecutive 1's are replaced with just the last 1 in that run). Thus, a base-3 number can be written in the form $f(a+b) - f(a) - f(b)$ if and only if its base-3 representation has only the digits 0 and 1. $\square$

Now it remains to count all base-3 numbers less than 2026 that are made with only the digits 0 and 1. The base-3 representation of 2026 is 2210001_3, so every string of 7 digits, each of which is either a 0 or a 1 (except for the zero string), can be written as $f(a+b) - f(a) - f(b)$, of which there are $2^7 - 1 = 127$.

5 Problem 5

Problem 5. Find the number of ways to fill each cell of a 10×10 grid with a 0 or a 1 such that every 2×2 subgrid contains exactly two 0s and exactly two 1s.

Proposed by Justin Lee

Answer: $\boxed{2046}$.

5.1 Solution

Let $a_{i,j}$ denote the entry of row i and column j of the grid. We make the following claim: if for some $1 \le i, j \le 9$ we have that $a_{i,j} = a_{i,j+1}$, then $a_{i,k} = 1 - a_{i+1,k}$ for all $1 \le k \le 10$. Indeed, this follows from applying the condition to the 2×2 subgrid consisting of $a_{i,j}, a_{i,j+1}, a_{i+1,j}, a_{i+1,j+1}$ to observe that $a_{i+1,j} = 1 - a_{i,j}$. Now, knowing the entries of row i as well as one entry of row $i+1$ uniquely determines the entries of row $i+1$: each entry must be given by $a_{i+1,k} = 1 - a_{i,k}$.

We now consider two cases: if row 1 contains two adjacent cells with equal values, then by the above paragraph, every other entry of the grid is uniquely determined: $a_{i,j} = a_{1,j}$ if i is odd and $a_{i,j} = 1 - a_{1,j}$ if i is even. Every such grid constructed in this manner satisfies the constraints in the problem, so the number of possible arrangements arising from this case equals the number of possibilities for the first row, which is $2^{10} - 2$ (as there are two arrangements for the first row that do *not* contain two adjacent cells with equal values).

If row 1 consists of $0101 \cdots 01$ or $1010 \cdots 10$, then inductively, we see that every row i must consist of $0101 \cdots 01$ or $1010 \cdots 10$. Moreover, every grid where each row consists of one of these two sequences of values is a grid that satisfies the constraints in the problem. Therefore, there are exactly 2^{10} grids in this case.

Combining, we have a total of $2^{10} - 2 + 2^{10} = 2046$ grids.

6 Problem 6

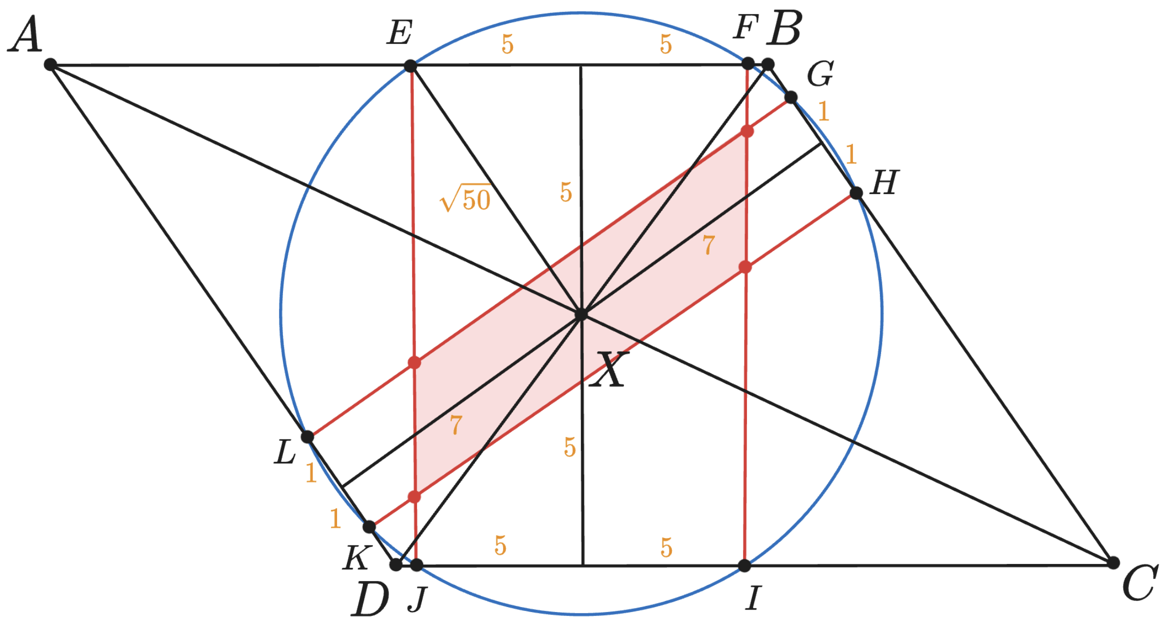
Problem 6. Two strips of paper with widths 10 and 14 overlap on a region that is a parallelogram with area 168. A point P lies inside this parallelogram such that the feet of the altitudes from P to the four sides of the parallelogram all lie on a circle with area 50π. The four possible points P which satisfy this form another parallelogram. What is its area?

Proposed by Vivian Loh

Answer: $\boxed{24}$.

6.1 Solution

Let the parallelogram be $ABCD$, where the height (when AB and CD are bases) is 10, and the height (when AD and BC are bases) is 14. Let the circle formed by feet of altitudes be $EFGHIJKL$, where E, F are its intersections with side AB, G, H its intersections with side BC, I, J its intersections with side CD, and K, L its intersections with side DA (as in the picture below). Suppose F, G, I, L are the feet of the altitudes from P to AB, BC, CD, and DA, respectively.



Now we claim that the center of the circle is the same as the center of the parallelogram, which we will call X. This is because the center of the circle is the intersection of the perpendicular bisectors of FI and GL, which is the center of the parallelogram. Thus, the diagram is fully symmetric about X.

Now, since the distance from X to sides AB and CD is 5 and the distance from X to sides BC and DA is 7, we know by the Pythagorean Theorem that $EF = JI = 2 \cdot \sqrt{50 - 5^2} = 2 \cdot 5$, and $GH = 2 \cdot \sqrt{50 - 7^2} = 2 \cdot 1$. The four possible choices of the point P are the red points in the picture – the pairwise intersections of $\{FI, EJ\}$ with $\{GL, HK\}$. The parallelogram with these four red points as corners has the same angle measures as the original parallelogram $ABCD$, but instead of heights 10 and 14 it has heights 10 and 2. Thus, its area is $\frac{10 \cdot 2}{10 \cdot 14} \cdot$ (area of $ABCD$), which is $\frac{168}{7} = 24$.

7 Problem 7

Problem 7. Let $a_0, a_1, \cdots, a_{10}$ be a strictly increasing sequence of positive integers such that $a_0 = 0$, $a_{10} = 67$, and the difference between every pair of consecutive terms is either 6 or 7. How many such sequences exist for which $\gcd(a_i, a_{i+1}) = 1$ for each $i \in \{1, 2, 3, \cdots, 8, 9\}$?

Proposed by Matthias Kim

Answer: $\boxed{12}$.

7.1 Solution

We split the problem into two conditions:

1. $a_0, a_1, \cdots, a_{10}$ is a strictly increasing sequence of positive integers such that $a_0 = 0$, $a_{10} = 67$, and the difference between every pair of consecutive terms is either 6 or 7.

2. $\gcd(a_i, a_{i+1}) = 1$ for each $i \in \{1, 2, 3, \cdots, 8, 9\}$

Let x be the number of $i \in \{0, 1, 2, \cdots, 8, 9\}$ for which $d_{i+1} = a_{i+1} - a_i = 6$, and let y be the number of such i in the set for which $a_{i+1} - a_i = 7$. Then, we have

$$\begin{cases} x + y = 10 \\ 6x + 7y = 67 \end{cases}$$

This system of equations yields $x = 3$ and $y = 7$. Therefore, the set of sequences $d_1, d_2, \cdots, d_9, d_{10}$ consisting of three 6's and seven 7's is in a natural, bijective correspondence with the set of sequences satisfying condition #1.

Hence, we constructively count the number of sequences (d_i) by starting with a sequence of seven 7's and inserting three 6's in a way so that the second condition is satisfied. By the Euclidean Algorithn,

$$\gcd(a_i, a_{i+1}) = 1 \Leftrightarrow \gcd(d_{i+1}, a_i) = 1$$

If $d_{i+1} = 7$, since 7 is prime, $\gcd(d_{i+1}, a_i) = 1$ is true if and only if $7 \nmid a_i$. $7 \mid a_i$ happens for $i \geq 1$ and only if $a_1 = 7$, $a_2 = 14$, or equivalently, $d_1 = d_2 = 7$. Suppose otherwise. First, if $a_1 = 6$, we have for $i \geq 2$, $a_i \equiv -1, -2, -3 \pmod 7$ since $d_i \equiv 0, -1 \pmod 7$, with $d_i \equiv -1 \pmod 7$ occuring only $y = 3$ times. Else, if $a_1 = 7$ and $d_2 = 6$, we similarly have $a_i \equiv -1, -2, -3 \pmod 7$ for $i \geq 2$.

Else, if $d_{i+1} = 6$, it is necessary and sufficient that $a_i \equiv 1, 5 \pmod 6$. Initially, when we begin with $7's$ sevens in $(d_n)_{n=1}^{10}$, these residues modulo 6 will appear in (a_n), in this order:

$$1, 2, 3, 4, 5, 0, 1$$

Critically, inserting $6's$ into (d_n) does not affect the order of the residues modulo 6 in (a_n). To satisfy the gcd condition for $d_{i+1} = 6$, we can only insert $d_{i+1} = a_{i+1} - a_i = 6$ between the blocks and at the ends

$$\boxed{1}\ \boxed{2345}\ \boxed{01}$$

Additionally, we cannot have $d_1 = d_2 = 6$ (two 6's in the beginning). To prevent $d_1 = d_2 = 7$, we must insert a 6 in the beginning or in between the blocks $\boxed{1}$ and $\boxed{2345}$. These conditions are sufficient and necessary for the gcd condition to be satisfied. Using stars and bars, there are

$$\binom{3+4-1}{4-1} = 20$$

ways to insert the 6's between blocks and at the ends. Accounting for additional restrictions, we must subtract

$$\overbrace{\binom{3+2-1}{2-1}}^{d_1=d_2=7} + \overbrace{4}^{d_1=d_2=6} = 8$$

from our total to yield 12 total sequences.

8 Problem 8

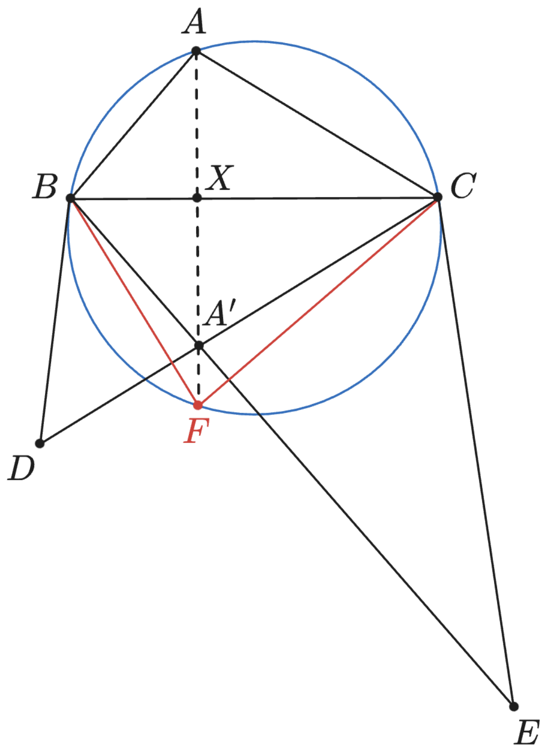
Problem 8. Triangle ABC has area 1. Points D and E are on the opposite side of line BC from A such that triangles BDC and CBE are both similar to triangle ABC, and have areas 2 and 4 respectively. The perpendicular line from B to CD intersects the perpendicular line from C to BE at point F. The length of AF can be expressed as $\frac{a}{b^{3/4}}$, for relatively prime positive integers a and b. Find $a + b$.

Proposed by Vivian Loh

Answer: $\boxed{18}$.

8.1 Solution

First, we will make some observations about angles: $\triangle BDC \sim \triangle ABC$ gives us $\angle ACB = \angle BCD$, and $\triangle ABC \sim \triangle CBE$ gives us $\angle ABC = \angle CBE$. So, the intersection of CD and BE is the reflection A' of A over BC. Point F is the intersection of altitudes in $\triangle A'BC$, so it is the orthocenter of $\triangle A'BC$. Thus, F is the second intersection of the line through A perpendicular to BC with the circumcircle (ABC).



Next, we will use the numbers 1, 2, and 4 to get some information about side lengths. The area ratios between $\triangle ABC$, $\triangle BDC$, and $\triangle CBE$ tell us that $AB : AC = 1 : \sqrt{2}$, and $AB : BC = 1 : 2$. Thus, $AB : AC : BC = 1 : \sqrt{2} : 2$. Let X be the intersection point of BC with the altitude AA'. Then by Power of a Point in the circumcircle (ABC), $XF \cdot XA = XB \cdot XC$. We can find the exact values of XA, XB, and XC given that we know $\triangle ABC$ has sides in the ratio $1 : \sqrt{2} : 2$ and total area 1, and then we can use Power of a Point to calculate XF.

Let us first create a triangle $\triangle A^*B^*C^*$ with sides of lengths $A^*B^* = 1$, $A^*C^* = \sqrt{2}$, and $B^*C^* = 2$, and then scale it appropriately to obtain $\triangle ABC$ with area 1. If X^* is the foot of the altitude from A^* to B^*C^*, then $X^*C^* = 2 - X^*B^*$. Then we know that $(X^*C^*)^2 - (X^*B^*)^2 = 4 - 4(X^*B^*)$ By the Pythagorean Theorem on $\triangle X^*A^*B^*$ and $\triangle X^*A^*C^*$, $(X^*C^*)^2 - (X^*B^*)^2 = (A^*C^*)^2 - (A^*B^*)^2 = 1$. So, $1 = 4 - 4(X^*B^*)$, implying that $X^*B^* = \frac{3}{4}$ and $X^*C^* = 2 - \frac{3}{4} = \frac{5}{4}$. By the Pythagorean Theorem, $X^*A^* = \sqrt{1 - (X^*B^*)^2} = \sqrt{1 - \left(\frac{3}{4}\right)^2} = \frac{\sqrt{7}}{4}$, so the total area of $\triangle A^*B^*C^*$ is $\frac{1}{2} \cdot 2 \cdot \frac{\sqrt{7}}{4} = \frac{\sqrt{7}}{4}$. Thus, the ratio of side lengths between $\triangle A^*B^*C^*$ and its counterpart $\triangle ABC$ is $1 : \sqrt{\frac{4}{\sqrt{7}}} = 1 : \frac{2}{\sqrt[4]{7}}$. This gives us the exact values of XA, XB, and XC:

$$XA = \frac{\sqrt{7}}{4} \cdot \frac{2}{\sqrt[4]{7}}$$
$$XB = \frac{3}{4} \cdot \frac{2}{\sqrt[4]{7}}$$
$$XC = \frac{5}{4} \cdot \frac{2}{\sqrt[4]{7}}.$$

Now, plugging in $XF = \frac{XB \cdot XC}{XA}$ from Power of a Point gives us

$$XF = \frac{15/\sqrt{7}}{4} \cdot \frac{2}{\sqrt[4]{7}}$$

so the total length of AF is

$$XA + XF = \left(\frac{\sqrt{7}}{4} + \frac{15/\sqrt{7}}{4}\right) \cdot \frac{2}{\sqrt[4]{7}}$$
$$= \frac{22/\sqrt{7}}{4} \cdot \frac{2}{\sqrt[4]{7}} = \frac{11}{2\sqrt{7}} \cdot \frac{2}{\sqrt[4]{7}} = \frac{11}{7^{3/4}}.$$

Thus, the answer is $11 + 7 = 18$. □

9 Problem 9

Problem 9. Find the smallest positive integer a for which there exist positive integers b and c such that

$$\frac{a}{b^2} + \frac{1}{c} = \frac{1}{2026}.$$

Proposed by Justin Lee

Answer: $\boxed{3}$.

9.1 Solution

Note that we can rewrite the equation as

$$\frac{c-2026}{c \cdot 2026} = \frac{1}{2026} - \frac{1}{c} = \frac{a}{b^2}$$

Note that we need to look no farther than $a \leq 1013$, since choosing $c = 2 \cdot 2026$ gives $\frac{1}{2 \cdot 2026} = \frac{a}{b^2} = \frac{1013}{2026^2}$. Now we consider two cases: first, if $1013 \nmid c$, then $\nu_{1013}(2026c) = 1$ and $\nu_{1013}(c - 2026) = 0$, so it follows that $\nu_{1013}(a) - 2\nu_{1013}(b) = -1 \implies \nu_{1013}(a) \geq 1$ for parity reasons. Then, $a \geq 1013$, which is no more optimal than what we have already found. In the case where $1013 \mid c$, let us write $c = 1013d$ so that

$$\frac{a}{b^2} = \frac{d-2}{2026d}$$

Likewise to the above, observe that if $1013 \nmid d$ and $1013 \nmid d - 2$, then $\nu_{1013}(a) \geq 1$. Thus, to find a more optimal value of a, we must have $d \equiv 0$ or $2 \mod 1013$. In the $1013 \mid d$ case, observe $\gcd(d - 2, 2026d) \mid 4$, so it follows that $a \geq (d-2)/\gcd(d-2, 2026d)$: in particular, a is at least the largest odd divisor of $d - 2$, and when d is even, $a \geq (d-2)/4$. Thus, one only needs to test $d \in \{1013, 2 \cdot 1013, 3 \cdot 1013, 4 \cdot 1013\}$ to find the most optimal value of a in this case, which arises from $d = 2026$ yielding $a = 506$.

In the $d \equiv 2 \mod 1013$ case, observe that $\gcd(d - 2, 2026d) \mid 4 \cdot 1013$. In particular, we deduce that when d is odd, $a \geq (d-2)/1013$, whereas when d is even, $a \geq (d-2)/(4 \cdot 1013)$. Testing the smallest values of d, we note that $d = 2028$ yields $\frac{a}{b^2} = \frac{1}{2028}$ for which $a = 3$ and $b = 78$ works. From the above inequalities, we deduce that no smaller value of a can be obtained, after checking that $d = 1015, 4054, 8106$ clearly do not yield smaller values of a.

10 Problem 10

Problem 10. A cube is suspended in the air with its eight vertices 1, 2, 3, 4, 5, 6, 7, and 8 units off of the ground. What is its side length?

Proposed by Vivian Loh

Answer: $\boxed{\sqrt{21}}$.

10.1 Algebraic Solution

Let d be the height function, so that $d(x)$ denotes the distance from a point x to the ground. Note that for x in space above the ground, $d(x)$ is linear. In particular, if v_1, v_2, v_3 denote the distinct vectors corresponding to the edges of the cube with starting point at the lowest vertex, then there is a well defined $\Delta d(v_i) = d(x + v_i) - d(x)$ which does not depend on the choice of x. Then for every vertex p of the cube, there is a unique choice of $a_1, a_2, a_3 \in \{0, 1\}$ such that $d(p) = 1 + a_1 d(v_1) + a_2 d(v_2) + a_3 d(v_3)$, since the vector with starting point at the vertex of height 1 and ending point at p is uniquely expressed as $a_1 v_1 + a_2 v_2 + a_3 v_3$. By inspection, we see that $\{d(v_1), d(v_2), d(v_3)\} = \{1, 2, 4\}$.

Now let us coordinatize our space so that the vertices of the cube are $\{0,s\}^3$, where s is the side length of the cube. Consider the plane $\mathcal{P}$ passing through the origin parallel to the ground: its equation is of the form $ax+by+cz=0$ for certain constants a,b,c. By the above, we know that the distances from vertices $(s,0,0)$, $(0,s,0)$, and $(0,0,s)$ to $\mathcal{P}$ are $1,2,4$, in, WLOG, this order. We deduce, by the distance formula, that

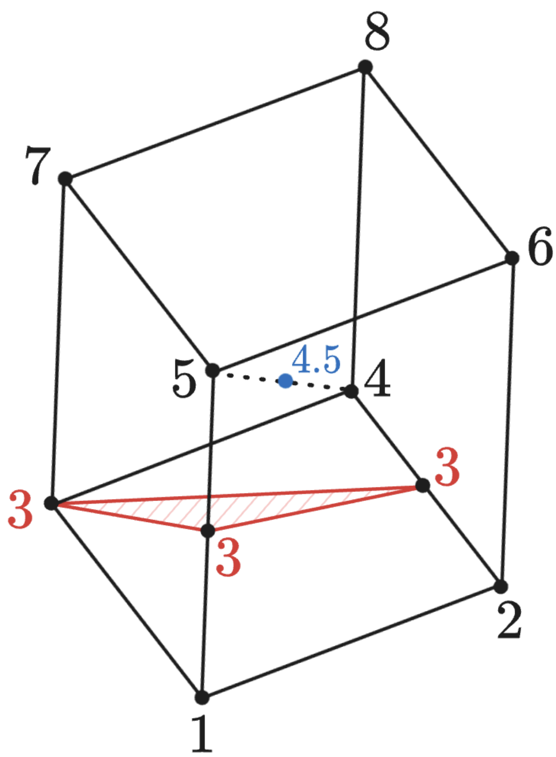
$$1=\frac{as}{\sqrt{a^2+b^2+c^2}} \qquad 2=\frac{bs}{\sqrt{a^2+b^2+c^2}} \qquad 4=\frac{cs}{\sqrt{a^2+b^2+c^2}}$$

and summing the squares of these equations yields $s^2=1^2+2^2+4^2 \implies s=\sqrt{21}$.

10.2 Geometric Solution

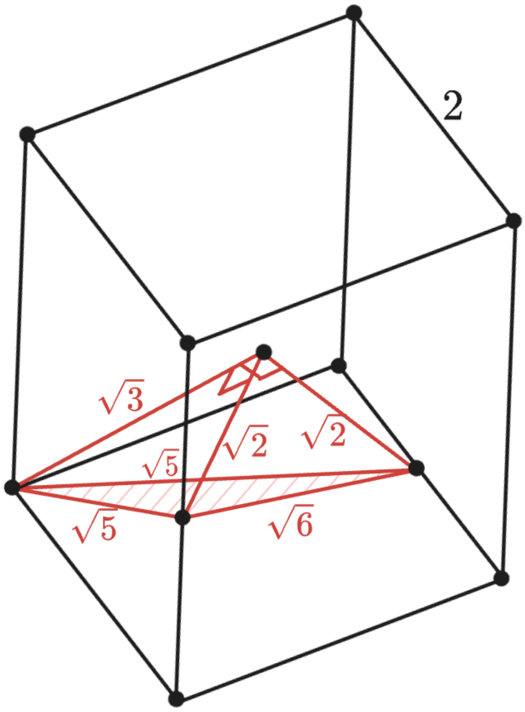
One initial observation is that opposite corners of the cube must have heights averaging to the height of the cube's center, so the vertices with heights $\{1,8\}$, $\{2,7\}$, $\{3,6\}$, $\{4,5\}$ must be pairs of opposites. Furthermore, on every face, each pair of opposite vertices must have heights averaging to the same value. The three vertices adjacent to the 1 must have heights 2, 3, and 5, because $\{1,2,4\}$ is the only possible set $\{a,b,c\}$ such that $\{1,1+a,1+b,1+c,1+a+b,1+a+c,1+b+c,1+a+b+c\}=\{1,2,3,4,5,6,7,8\}$.

Intuitively, we want to determine the "tilt" of the cube – what angle the cube is tilted at. Let us label many points with their heights off the ground, including the midpoints of two edges and the center of the cube. (The midpoint of the $1-5$ edge must be labeled with "3", and same with the midpoint of the $2-4$ edge.)



Thus, the plane through the three points labeled "3" (one vertex and two midpoints of sides) must be horizontal, and the distance from the center of the cube to this plane must be 1.5 (since the center of the cube is 4.5 units off the ground). Let this plane be called $\mathcal{P}$.

To determine the side length of the cube, we will determine the distance from the center of a **cube of side length 2** (for the sake of nice numbers) to its analogous plane $\mathcal{P}$, so that we may find the scale factor by which to transform the cube of side length 2 in order to obtain the actual cube. Consider the following tetrahedron connecting the center of the cube with the three vertices labeled "3":



By the Pythagorean Theorem, we find that the side lengths of this tetrahedron are $\sqrt{2}$, $\sqrt{2}$, $\sqrt{6}$, $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{5}$ as shown in the diagram. Furthermore, two of the faces of the tetrahedron are right triangles (with side lengths $\sqrt{2}, \sqrt{3}, \sqrt{5}$) and the two others are isosceles (with side lengths $\sqrt{2}, \sqrt{2}, \sqrt{6}$ and $\sqrt{5}, \sqrt{5}, \sqrt{6}$). Note that the ratio (tetrahedron height when placed flat on the $\sqrt{2}-\sqrt{2}-\sqrt{6}$ face) : (tetrahedron height when placed flat on the $\sqrt{5}-\sqrt{5}-\sqrt{6}$ face) is equal to (area of the $\sqrt{5}-\sqrt{5}-\sqrt{6}$ face) : (area of the $\sqrt{2}-\sqrt{2}-\sqrt{6}$ face). From the right angles, we know that the height of the tetrahedron when placed flat on the $\sqrt{2}-\sqrt{2}-\sqrt{6}$ face is $\sqrt{3}$, and the Pythagorean Theorem gives us that the areas of ($\sqrt{2}-\sqrt{2}-\sqrt{6}$ face) and ($\sqrt{5}-\sqrt{5}-\sqrt{6}$ face) are $\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{6}}{2}$ and $\frac{\sqrt{14}}{2}\cdot\frac{\sqrt{6}}{2}$, respectively. So, we have the equation

$$\sqrt{3} : (\text{height of tetrahedron when placed flat on } \sqrt{5}-\sqrt{5}-\sqrt{6} \text{ face}) = \frac{\sqrt{14}}{2}\cdot\frac{\sqrt{6}}{2} : \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{6}}{2},$$

therefore the desired height is $\frac{\sqrt{3}}{\sqrt{7}}$.

Remember that this was for cube of side length 2, so the scale factor that we must multiply the side lengths through by to obtain the desired cube is $1.5 \div \frac{\sqrt{3}}{\sqrt{7}}$. Thus, the side length of the actual cube is $2 \cdot 1.5 \cdot \frac{\sqrt{3}}{\sqrt{7}} = \sqrt{21}$.

11 Problem 11

Problem 11. Suppose the polynomial $x^4 + ax^3 + 69x^2 + bx + 100$ has real coefficients and roots z_1, z_2, z_3, and z_4, none of which are real. The possible values of $|z_1| + |z_2| + |z_3| + |z_4|$ form an open interval (m, M). Compute m.

Proposed by Justin Lee

Answer: $\boxed{14}$.

11.1 Solution

Since the roots must come in conjugate pairs, we can let them be $\{c+di, c-di, e+fi, e-fi\}$ for some real $\{c, d, e, f\}$. Then expanding the quadratic and constant coefficients, we obtain

$$69 = 4ec + c^2 + d^2 + e^2 + f^2 \qquad 100 = (c^2+d^2)(e^2+f^2)$$

Note that $|c| < \sqrt{c^2+d^2}$ and similarly $|e| < \sqrt{e^2+f^2}$, so if we let $\sqrt{c^2+d^2} = \alpha$, and $\sqrt{e^2+f^2} = \beta$, then we have

$$69 < 4\alpha\beta + \alpha^2 + \beta^2 \qquad 100 = \alpha^2\beta^2 ==> \alpha\beta = 10$$

Now, the desired quantity, $|z_1| + |z_2| + |z_3| + |z_4|$, is precisely $2(\alpha+\beta)$, and we have

$$(\alpha+\beta)^2 = \alpha^2 + \beta^2 + 4\alpha\beta - 2\alpha\beta > 69 - 40$$

implying $\alpha + \beta > 7$, so $m = 14$. Indeed, $\alpha + \beta$ can be arbitrarily close to 7 when the inequalities $|c| < \sqrt{c^2+d^2}$ and $|e| < \sqrt{e^2+f^2}$ become arbitrarily sharp, which is achieved when $d \to 0$ and $f \to 0$.

12 Problem 12

Problem 12. Three people are each initially given the number 0. Every hour, they each simultaneously flip a fair coin: for each person, if their coin lands as heads, they increase their number by one; otherwise, they don't change their number. What is the expected number of hours until the first occurrence of the three numbers all being distinct?

Proposed by Justin Lee

Answer: $\boxed{\dfrac{10+4\sqrt{2}}{3}}$.

12.1 Solution

For each integer $n \geq 0$, let $[n]$ denote the state where two of the three people have the same number, and the third person has a number which is either n larger or n smaller than the other two peoples'. In particular, the starting state is $[0]$. Now, from each state, we obtain the following probabilities of the state after one move:

- For $[0]$, we have a $\frac{3}{4}$ chance of arriving at state $[1]$ and a $\frac{1}{4}$ chance of staying at state $[0]$.
- For $[1]$, we have a $\frac{1}{8}$ chance of arriving at state $[0]$, a $\frac{1}{2}$ chance of staying at state $[1]$, a $\frac{1}{8}$ chance of arriving at state $[2]$, and a $\frac{1}{4}$ chance of the process terminating (via all numbers being distinct).
- For $[n]$ with $n \geq 2$, we have a $\frac{1}{8}$ chance of arriving at state $[n-1]$, a $\frac{1}{4}$ chance of staying at state $[n]$, a $\frac{1}{8}$ chance of arriving at state $[n+1]$, and a $\frac{1}{2}$ chance of the process terminating.

Let us first show that starting at any state $[n]$, the expected value of the number of hours until the process stops is uniformly bounded for all n. Let p_k denote the probability that (starting at state n), the process does *not* terminate after k steps. Then, the expected value of the number of hours until the process stops is $p_1 + p_2 + p_3 + \cdots$. However, note that $p_{k+2} \leq \frac{13}{16} p_k$, since regardless of the state after k steps, there is at least a $\frac{3}{16}$ probability that the process stops within two more steps. Therefore, the sum $p_1 + p_2 + \cdots$ is convergent, so the expected value is bounded.

Now let a_n denote the expected number of hours until the process stops if the starting state is $[n]$. We obtain the following relations:

- $a_n = 1 + \frac{1}{8}a_{n-1} + \frac{1}{4}a_n + \frac{1}{8}a_{n+1} \implies a_n = \frac{4}{3} + \frac{1}{6}a_{n-1} + \frac{1}{6}a_{n+1}$ when $n \geq 2$
- $a_1 = 1 + \frac{1}{2}a_1 + \frac{1}{8}a_0 + \frac{1}{8}a_2$
- $a_0 = 1 + \frac{1}{4}a_0 + \frac{3}{4}a_1$

The recurrence relation for $n \geq 2$ can be written as $(a_n - 2) = \frac{1}{6}(a_{n-1} - 2) + \frac{1}{6}(a_{n+1} - 2)$, from which we obtain that $a_n - 2 = c_1(3+2\sqrt{2})^n + c_2(3-2\sqrt{2})^n$ for certain constants c_1, c_2. However, in order for a_n to be bounded, we must have $c_1 = 0$. Hence, for $n \geq 1$, we have $a_n - 2 = c(3-2\sqrt{2})^n$, so in particular, we have $a_2 - 2 = (3-2\sqrt{2})(a_1 - 2)$. Combining this equation with the last two bullet points yields $a_0 = \frac{10+4\sqrt{2}}{3}$.