# Team Round

CHMMC 2016

November 20, 2016

**Problem 1.** Let $a_n$ be the $n$th positive integer such that when $n$ is written in base 3, the sum of the digits of $n$ is divisible by 3. For example, $a_1 = 5$ because $5 = 12_3$. Compute $a_{2016}$.

*Solution* 1. $\boxed{6049}$. The observation to make is that for three consecutive numbers $3k, 3k+1, 3k+2$, exactly one number will have a base 3 representation that has digital sum divisible by 3 because the digital sums of these three numbers are distinct in mod 3. In particular, $a_n \in (3n, 3n+1, 3n+2)$. It remains to check the triple $(6048, 6049, 6050)$ to see which number has the correct base 3 representation, and $6049 = 22022001_3$.

**Problem 2.** Consider the $5 \times 5$ grid $\mathbb{Z}_5^2 = \{(a, b) : 0 \leq a, b \leq 4\}$. Say that two points $(a, b), (x, y)$ are adjacent if $a - x \equiv -1, 0, 1 \pmod 5$ and $b - y \equiv -1, 0, 1 \pmod 5$ . For example, in the diagram, all of the squares marked with $\cdot$ are adjacent to the square marked with $\times$.

| | | | | |
|---|---|---|---|---|
| $\times$ | $\cdot$ | | | $\cdot$ |
| $\cdot$ | $\cdot$ | | | $\cdot$ |
| | | | | |
| | | | | |
| $\cdot$ | $\cdot$ | | | $\cdot$ |

What is the largest number of $\times$ that can be placed on the grid such that no two are adjacent?

*Solution* 2. $\boxed{5}$. First, see that we can place 5 $\times$ on the grid. Start with one $\times$ in any location. Fix a direction, make one knight's move away in that direction and place another $\times$. Repeat three times. The result looks like this:

| | | | | |
|---|---|---|---|---|
| $\times$ | $\cdot$ | | | $\cdot$ |
| $\cdot$ | $\cdot$ | $\times$ | | $\cdot$ |
| | | | | $\times$ |
| | $\times$ | | | |
| $\cdot$ | $\cdot$ | | $\times$ | $\cdot$ |

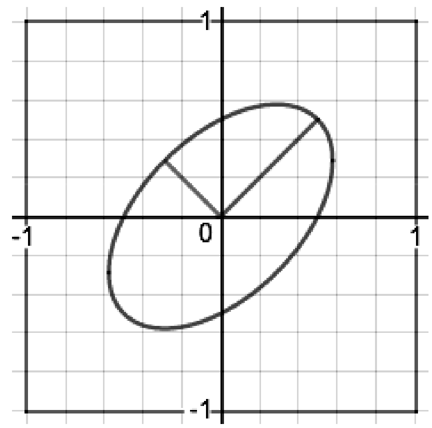
Next, we see that we can't have more than 5. Suppose we did. Then by pigeonhole, at least one column has at least two $\times$ in it. In the remainder, we show that if some column has two $\times$, then there are at most 4 $\times$ in the whole grid.

Notice that a column may not have more than two $\times$ in it. Furthermore, if one column has two $\times$, then the adjacent columns must be empty. Then there are two columns not adjacent to the column which has two $\times$ in it. If either column has two $\times$, then the other is empty. Therefore, there are at most two $\times$ among them.

**Problem 3.** For a positive integer $m$, let $f(m)$ be the number of positive integers $q \leq m$ such that $\frac{q^2-4}{m}$ is an integer. How many positive square-free integers $m < 2016$ satisfy $f(m) \geq 16$?

*Solution* 3. Note that $q^2 \equiv 4 \pmod{p_1 \cdots p_k}$ if and only if $q \equiv \pm 2 \pmod{p_i}$ for every $1 \le i \le k$, where the $p_i$ are distinct primes. Then letting $m = p_1 \cdots p_k$, we see that $f(m) = 2^k$ if $2 \nmid m$ and $f(m) = 2^{k-1}$ if $2 \mid m$. Thus, $f(m) \ge 16$ if and only if $m$ is divisible by at least 4 odd primes. Some quick enumeration shows that the only such values for $m$ less than 2016 are $3 \cdot 5 \cdot 7 \cdot 11$, $3 \cdot 5 \cdot 7 \cdot 13$, $3 \cdot 5 \cdot 7 \cdot 17$, and $3 \cdot 5 \cdot 7 \cdot 19$. Thus there are $\boxed{4}$ possibilities.

**Problem 4.** Line segments $m$ and $n$ both have length 2 and bisect each other at an angle of $60°$, as shown. A point $X$ is placed at uniform random position along $n$, and a point $Y$ is placed at a uniform random position along $m$. Find the probability that the distance between $X$ and $Y$ is less than $\frac{1}{2}$.




*Solution* 4. Let $x$ and $y$ be variables chosen randomly and uniformly from the range $[-1, 1]$ representing the positions of $X$ and $Y$ along $n$ and $m$, respectively, with a value of 1 for each representing the left-most point on their line-segments in the diagram above. Then letting $z$ be the distance between $X$ and $Y$, by the law of cosines and a simple case analysis,

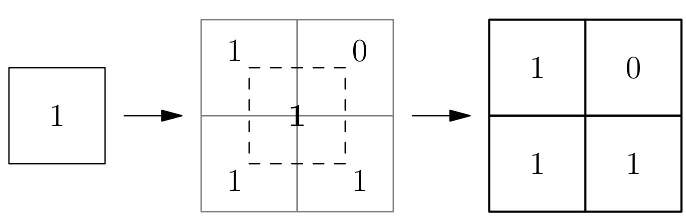
$$x^2 + y^2 = z^2 + 2xy\cos(60°) = z^2 + xy,$$

so our condition is equivalent to $x^2 - xy + y^2 < \frac{1}{4}$. We can rewrite this as $(x+y)^2 + 3(x-y)^2 < 1$, which shows that the region in the $x, y$ plane satisfying our condition is an ellipse with semi-major/minor axes along the directions $(1, 1)$ and $(1, -1)$, as shown.

We now solve for the lengths of these axes. If $x + y = 0$, then $x = -y$, so $3(2x)^2 < 1 \Rightarrow x^2 < \frac{1}{12} \Rightarrow |x| < \frac{1}{\sqrt{12}}$. Thus one of the semi-axes goes from $(0,0)$ to $(\frac{1}{\sqrt{12}}, \frac{-1}{\sqrt{12}})$, hence has length $\frac{1}{\sqrt{6}}$. Similarly, if $x - y = 0$, then $(2x)^2 < 1 \Rightarrow |x| < \frac{1}{2}$, so one of the semi-axes goes from $(0,0)$ to $(\frac{1}{2}, \frac{1}{2})$, hence has length $\frac{1}{\sqrt{2}}$. The area of the ellipse is thus $\pi \frac{1}{\sqrt{6}} \frac{1}{\sqrt{2}} = \frac{\pi}{2\sqrt{3}}$. The total region from which we are choosing $(x, y)$ is a square of side length 2, thus of area 4. Our answer is thus $\frac{\pi}{2\sqrt{3}} \cdot \frac{1}{4} = \boxed{\frac{\pi}{8\sqrt{3}}}$.

**Problem 5.** Given a triangle $ABC$, let $D$ be a point on segment $BC$. Construct the circumcircle $\omega$ of triangle $ABD$ and point $E$ on $\omega$ such that $CE$ is tangent to $\omega$ and $A, E$ are on opposite sides of $BC$(as shown in diagram). If $\angle CAD = \angle ECD$ and $AC = 12, AB = 7$, find $AE$.

*Solution* 5.



We claim that $\triangle CAE \sim \triangle EAB$, from which we conclude that

$$\frac{AC}{AE} = \frac{AE}{AB} \implies AE = \sqrt{AC \cdot AB} = \boxed{2\sqrt{21}}$$

.

To see the similarity, note that properties of arcs and tangents tell us that $\angle ABE = \angle AEC$ (the first set of angle equalities), $\angle CED = \angle EAD$, and $\angle BAE = \angle BDE$. Thus

$$\angle BAE = \angle BDE = \angle ECD + \angle CED = \angle CAD + \angle EAD = \angle CAE.$$

This gives us the second set of angle equalities, from which we conclude that $\triangle CAE \sim \triangle EAB$.

**Problem 6.** For any nonempty set of integers $X$, define the function

$$f(X) = \frac{(-1)^{|X|}}{\left(\prod_{k \in X} k\right)^2}$$

where $|X|$ denotes the number of elements in $X$.

Consider the set $S = \{2, 3, \ldots, 13\}$. Note that 1 is **not** an element of $S$.

Compute

$$\sum_{\substack{T \subseteq S \\ T \neq \emptyset}} f(T).$$

where the sum is taken over all nonempty subsets $T$ of $S$.

*Solution* 6. If we add 1 to the given sum, the resulting expression is just the expanded version of the product

$$\prod_{j \in S} \left(1 - \frac{1}{j^2}\right) = \prod_{j=2}^{13} \left(1 - \frac{1}{j^2}\right).$$

We may rewrite the product as

$$\prod_{j=2}^{13} \frac{j^2 - 1}{j^2}$$

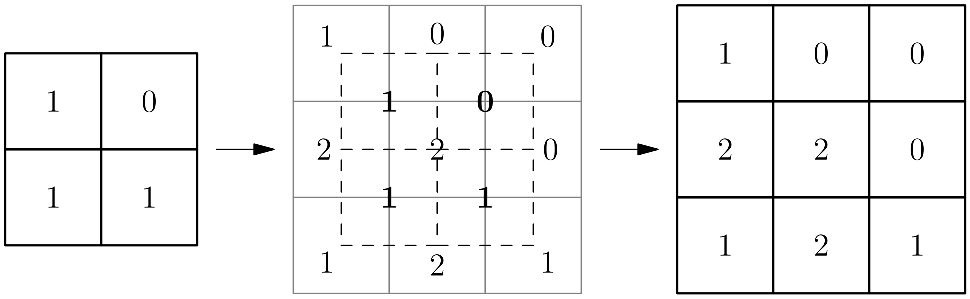
and then apply difference of squares to get that the product is equal to

$$\begin{aligned}
\prod_{j=2}^{13} \frac{j^2 - 1}{j^2} &= \prod_{j=2}^{13} \frac{(j-1)(j+1)}{j^2} \\
&= \prod_{j=2}^{13} \frac{j-1}{j} \cdot \prod_{j=2}^{13} \frac{j+1}{j} \\
&= \frac{1}{13} \cdot \frac{14}{2} = \frac{7}{13}
\end{aligned}$$

where in the transition between the second and third lines we used the fact that each individual product telescoped. Thus the sum is equal to $7/13 - 1 = \boxed{-6/13}$.

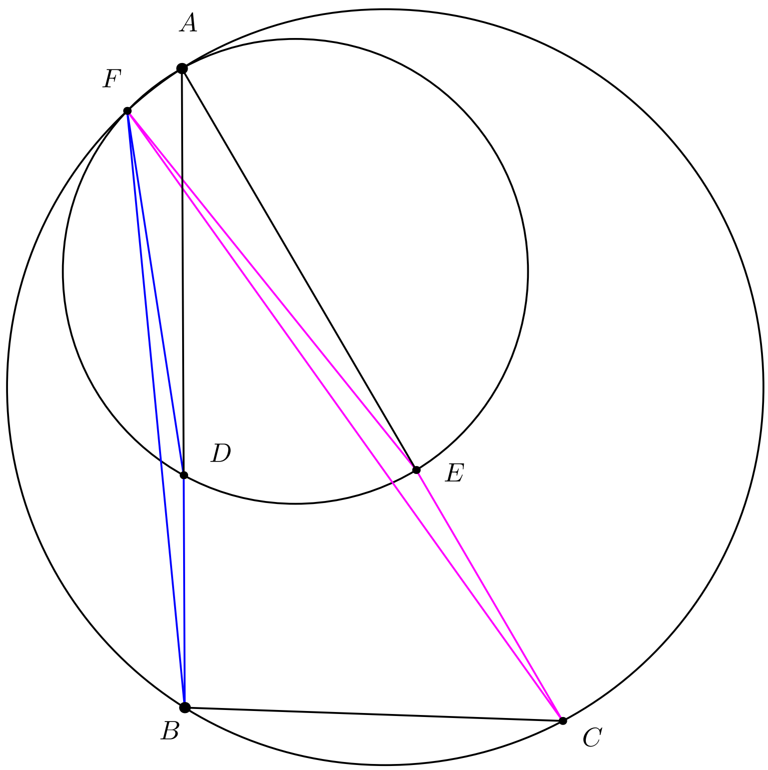
**Problem 7.** Consider constructing a tower of tables of numbers as follows. The first table is a one by one array containing the single number 1.

The second table is a two by two array formed underneath the first table and built as followed. For each entry, we look at the terms in the previous table that are directly up and to the left, up and to the right, and down and to the right of the entry, and we fill that entry with the sum of the numbers occurring there. If there happens to be no term at a particular location, it contributes a value of zero to the sum.




The diagram above shows how we compute the second table from the first.

The diagram below shows how to then compute the third table from the second.




For example, the entry in the middle row and middle column of the third table is equal the sum of the top left entry 1, the top right entry 0, and the bottom right entry 1 from the second table, which is just 2.

Similarly, to compute the bottom rightmost entry in the third table, we look above it to the left and see that the entry in the second table's bottom rightmost entry is 1. There are no entries from the second table above it and to the right or below it and to the right, so we just take this entry in the third table to be 1.

We continue constructing the tower by making more tables from the previous tables. Find the entry in the third (from the bottom) row of the third (from the left) column of the tenth table in this resulting tower.

*Solution* 7. Let $a_{j,k}^{(i)}$ denote the entry in the $j^{\text{th}}$ row and $k^{\text{th}}$ column of the $i^{\text{th}}$ table in the tower, where the row index $j$ starts at zero (for the bottom row), the column index $k$ starts at zero (for the leftmost column), and the index $i$ starts at one.

To each table in the tower, we can associate the polynomial

$$f_i(x,y) = \sum_{j=0}^{i-1}\sum_{k=0}^{i-1} a_{j,k}^{(i)} x^j y^k.$$

The recursion we use to build up the $(i+1)^{\text{th}}$ table from the $i^{\text{th}}$ table corresponds to the polynomial recurrence

$$f_{i+1}(x,y) = (1+x+y)\cdot f_i(x,y).$$

Since $f_1(x,y) = 1$, it follows that

$$f_i(x,y) = (1+x+y)^{i-1}.$$

Hence when the problem is asking us to find the entry in the third (from the bottom) row of the third (from the left) column of the tenth table in the tower, it is really asking us to compute the coefficient of $x^2y^2$ in $f_{10}(x,y)$. There are multiple ways find the answer from this point. If we use the binomial theorem, we can get that the answer is

$$\binom{4}{2}\binom{9}{4} = 6\cdot(9\cdot 2\cdot 7) = \boxed{756}.$$

**Problem 8.** Let $n$ be a positive integer. If $S$ is a nonempty set of positive integers, then we say $S$ is *$n$-complete* if all elements of $S$ are divisors of $n$, and if $d_1$ and $d_2$ are any elements of $S$, then $n/d_1$ and $\gcd(d_1, d_2)$ are in $S$. How many 2310-complete sets are there?

*Solution* 8. Factor $2310 = 2\cdot 3\cdot 5\cdot 7\cdot 11$. Recall that a partition of a set $T$ is a collection of disjoint nonempty subsets of $T$ whose union is all of $T$. We prove that there is a 1-1 correspondence between 2310-complete sets and partitions of $F = \{2, 3, 5, 7, 11\}$, as follows.

To each 2310-complete set, we can associate a nonempty collection of subsets of $F$, by replacing each number in the set with its set of prime factors. This collection is closed under intersections and complements in $F$ by definition of 2310-completeness. Conversely, any collection having these properties corresponds to a 2310-complete set, by replacing each subset with the product of its elements.

Given a collection of the above form, let $\mathcal{P}$ be its collection of minimal nonempty sets, i.e., sets which do not contain any smaller nonempty sets in the collection. Then $\mathcal{P}$ is a partition of $F$: any two sets in $\mathcal{P}$ are disjoint and nonempty by definition; and any element of $F$ is contained in some set in the collection because the collection is closed under complement, and then the intersection of all sets containing that element is in $\mathcal{P}$.

Conversely, given a partition, we can form the collection of all unions of sets in the partition, which is closed under intersections and complements. Now any collection closed under intersections and complements is also closed under unions. Then easily the two constructions above are inverse to each other, so each collection of the above form corresponds to a unique partition of $F$, proving the claim.

The answer is now the number of partitions of a 5-element set. This can be computed by case analysis of all possible sizes for the various sets in the partition:

$$5 : 1 \qquad 4+1 : \binom{5}{4} = 5 \qquad 3+2 : \binom{5}{3} = 10$$

$$3+1+1 : \binom{5}{3} = 10 \qquad 2+2+1 : \frac{1}{2}\binom{5}{2,2,1} = 15 \qquad 2+1+1+1 : \binom{5}{2} = 10$$

$$1+1+1+1+1 : 1$$

Then the answer is $1+5+10+10+15+10+1 = \boxed{52}$.

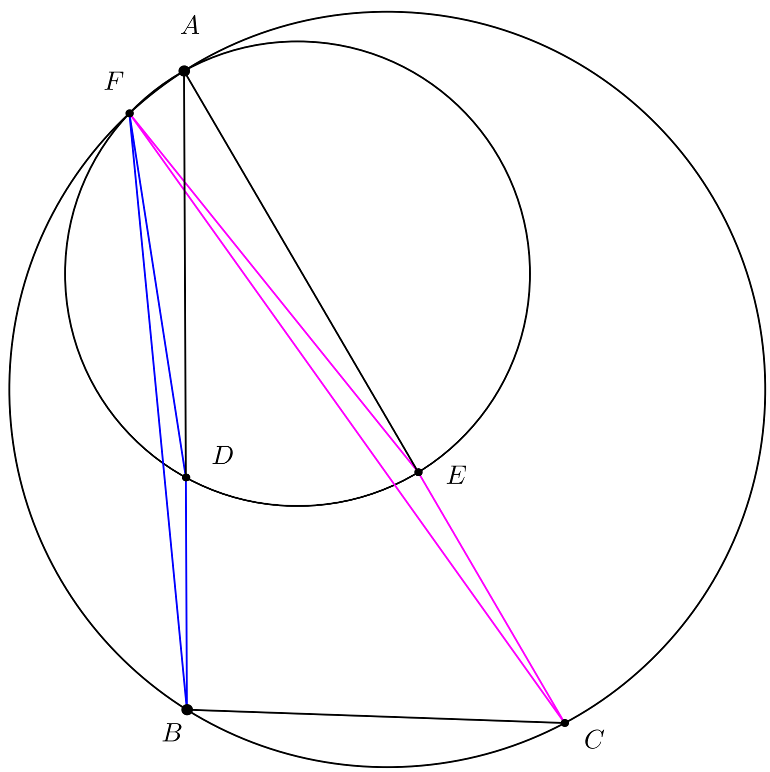
**Problem 9.** Find the sum of all 3-digit numbers whose digits, when read from left to right, form a strictly increasing sequence. (Numbers with a leading zero, e.g. "087" or "002", are not counted as having 3 digits.)

*Solution* 9. We write our numbers as $abc$, where $a$, $b$, and $c$ are digits, and $1 \le a < b < c \le 9$. First, we find the sum of all the $c$'s. If $c = 9$, there are 8 remaining numbers from which to pick $a$ and $b$, and given any choice of two of those numbers, the smallest one must be $a$. We thus have $\binom{8}{2}$ such numbers. Similarly, for every $3 \le c \le 9$, (note that $c$ must be greater than 2), we have $\binom{c-1}{2}$ choices. The sum of all the $c$'s is thus

$$\sum_{c=3}^{9} c\binom{c-1}{2} = \sum_{c=3}^{9} \frac{c(c-1)(c-2)}{2} = 3\sum_{c=3}^{9}\binom{c}{3} = 3\left(\binom{4}{4} + \sum_{c=4}^{9}\binom{c}{3}\right) = 3\binom{10}{4} = 3 \cdot 210 = 630,$$

where we have used the recursion $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ repeatedly. Next, we find the sum of all the $b$'s. To make a number satisfying our conditions is to pick three distinct numbers from the range 1 to 9, inclusive, and to then write them down in ascending order, so we have $\binom{9}{3} = 84$ such numbers. By symmetry, the average value of $b$ across all those numbers must be 5, so the sum of all the $b$'s is $5 \cdot 84 = 420$. Finally, since the $a$'s and the $c$'s should be symmetrically distributed about the number 5, the sum of the $a$'s must be $420 - (630 - 420) = 210$. Since the $a$'s represent hundreds, the $b$'s represent tens, and the $c$'s represent ones, our total sum is $21000 + 4200 + 630 = \boxed{25830}$.

**Problem 10.** Let $ABC$ be a triangle with circumcircle $\omega$ such that $AB = 11$, $AC = 13$, and $\angle A = 30^\circ$. Points $D$ and $E$ are on segments $AB$ and $AC$ respectively such that $AD = 7$ and $AE = 8$. There exists a unique point $F \neq A$ on minor arc $AB$ of $\omega$ such that $\angle FDA = \angle FEA$. Compute $FA^2$.




*Solution* 10.

$\angle FDA = \angle FEA$, so quadrilateral $AFDE$ is cyclic. By properties of arcs, $\angle FBA = \angle FCA$. Also, easily $\angle FDB = 180^\circ - \angle FDA = 180^\circ - \angle FEA = \angle FEC$. Then by AA, triangles $FBD$ and $FCE$ are similar, so $\frac{FB}{FC} = \frac{BD}{CE} = \frac{AB-AD}{AC-AE} = \frac{11-7}{13-8} = \frac{4}{5}$, so there exists a real number $x$ such that $FB = 4x$ and $FC = 5x$. By properties of arcs and assumption, $\angle BFC = \angle BAC = 30^\circ$. Then by the Law of Cosines,

$$BC^2 = FB^2 + FC^2 - 2FB \cdot FC \cos\angle BFC = (4x)^2 + (5x)^2 - 2(4x)(5x)\left(\frac{\sqrt{3}}{2}\right) = (41 - 20\sqrt{3})x^2,$$

so $x^2 = \frac{BC^2}{41-20\sqrt{3}}$. By Ptolemy's Theorem, $FA \cdot BC + FB \cdot AC = FC \cdot AB$, so

$$FA = \frac{FC \cdot AB - FB \cdot AC}{BC} = \frac{5x \cdot 11 - 4x \cdot 13}{BC} = \frac{3x}{BC}.$$

Squaring gives

$$FA^2 = \frac{9x^2}{BC^2} = \frac{9}{41 - 20\sqrt{3}} = \frac{9(41 + 20\sqrt{3})}{41^2 - 3 \cdot 20^2} = \boxed{\frac{369 + 180\sqrt{3}}{481}}.$$