Individual Round 2024-2025

Problem 1. Caltech Hall has 9 floors. An elevator starts at floor 1 and goes up to floor 9, stopping at every floor. The eight numbers 2 through 9 are randomly handed out on slips of paper, two to each person. Each person gets on the elevator at their lower number and gets off at their higher number. What is the probability that everyone is in the elevator at once?

Problem 2. Find the smallest positive integer n such that $n \cdot 2^{17} + 1$ has a divisor d such that $d+1$ is divisible by 2^{12}.

Problem 3. Find the largest positive integer $n \neq 2025$ such that $nx - \lfloor \sqrt{n} \lfloor \sqrt{n} \lfloor x \rfloor \rfloor \rfloor \leq 2025$ holds for all real x. (Here $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.)

Problem 4. In rectangle $ABCD$, points E and F are chosen on sides BC and CD respectively such that EF bisects $\angle CFA$ and $\angle AEF = 90^\circ$. Given that $CF = 9$ and $FD = 7$, what is the area of triangle AEF?

Problem 5. Find the number of odd positive integers $n < 1000$ such that if S_n denotes the set of divisors of n, then exactly half of the elements of S_n are divisible by 3, exactly a third of the elements of S_n leave a remainder of 1 upon dividing by 3, and exactly a sixth of the elements of S_n leave a remainder of 2 upon dividing by 3.

Problem 6. Determine the value of

$$\prod_{n=1}^{\infty} \left(\frac{1 + 2\cos\left(\frac{\pi}{3^n}\right)}{3} \right) = \frac{1+2\cos\left(\frac{\pi}{3}\right)}{3} \cdot \frac{1+2\cos\left(\frac{\pi}{9}\right)}{3} \cdot \frac{1+2\cos\left(\frac{\pi}{27}\right)}{3} \cdots .$$

Problem 7. On an infinite lattice grid, a circle with radius $0 < r < 10$ is drawn, centered at the origin. Each grid square is shaded if and only if the boundary of the circle passes through its interior (not only a corner or side). Let N be the number of grid squares that are shaded. How many possible values of N are there?

Problem 8. Points A, B, and C lie on a line in that order, and point D is such that $\angle DBC = 60^\circ$ and $\angle ADB + \angle ADC = 150^\circ$. Point E is on the circumcircle of $\triangle BCD$ such that line BE is tangent to the circumcircle of $\triangle ABD$. If $DE = 4$ and $CE = 3$, what is BE?

Problem 9. Let $\circ$ be a binary operation on the real numbers satisfying the properties

$$(ab) \circ (ac) = |a|(b \circ c) \qquad \text{for all real numbers } a, b, c$$

and

$$(a \circ b) \circ (c \circ d) = (a+b) \circ (c+d) \qquad \text{for all real numbers } a, b, c, d \text{ satisfying } ab + cd = 0$$

Find the product of all possible nonzero values of $20 \circ 25$.

Problem 10. Hannah has a 2024×2025 rectangle in the coordinate plane, with sides parallel to the axes. She makes a cut from one side to another side which only goes down and/or right along grid lines. Then she puts the two pieces together, possibly with rotations and/or reflections without overlaps or gaps, to form a new rectangle which is not congruent to the original. How many possible new rectangles can she produce? (An $a \times b$ rectangle is considered the same as a $b \times a$ rectangle.)

Problem 11. In triangle $\triangle ABC$, $\angle BAC = 90^\circ$. D is the foot of the altitude from A to BC. Point P is inside $\triangle ABC$ such that $\angle PAC = \angle PCB$. Points Q and R are inside $\triangle ABD$ and $\triangle ACD$, respectively, such that $\triangle PAC \sim \triangle QDA \sim \triangle RDC$. If $\triangle PQR$ is equilateral, what is $\frac{AB}{AC}$?

Problem 12. Suppose x, y, and z are complex numbers satisfying

$$|xyz| = 12$$
$$|x\bar{y}+z| = \frac{15\sqrt{3}}{2}$$
$$|y\bar{z}+x| = 3\sqrt{6}$$

Compute the minimum possible value of $|z\bar{x}+y|$.

(Here, $\bar{w}$ denotes the complex conjugate of w and $|w|$ denotes the norm of w, where $|a+bi| = \sqrt{a^2+b^2}$.)