Fall 2012 Caltech-Harvey Mudd Math Competition

November 17, 2012

Tiebreaker Round

The tiebreaker round will be similar to the ARML tiebreaker round. The top students with the same score on the individual round will be given a question to be solved in ten minutes. The students will be able to submit an answer only once, and they will be ranked according to the time when they submit a correct answer.

TBR1. Let $[n] = \{1, 2, 3, \ldots, n\}$ and for any set S, let $P(S)$ be the set of non-empty subsets of S. What is the last digit of $|P(P([2013]))|$?

TBR2. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(y)) = f(x + y) + y,$$

for all $x, y \in \mathbb{R}$. No proof is required for this problem.

TBR3. For a positive integer n, let $\sigma(n)$ be the sum of the divisors of n (for example, $\sigma(10) = 1 + 2 + 5 + 10 = 18$). For how many $n \in \{1, 2, \ldots, 100\}$ do we have $\sigma(n) < n + \sqrt{n}$?

TBR4. A lattice point $(x, y, z) \in \mathbb{Z}^3$ can be *seen from the origin* if the line from the origin does not contain any other lattice point (x', y', z') with $(x')^2 + (y')^2 + (z')^2 < x^2 + y^2 + z^2$. Let p be the probability that a randomly selected point on the cubic lattice $\mathbb{Z}^3$ can be seen from the origin. Given that

$$\frac{1}{p} = \sum_{n=i}^{\infty} \frac{k}{n^s}$$

for some integers i, k, and s, find i, k and s.