Individuals Tiebreaker 2024-2025 Solutions

Problem 1. There are four switches in a row, two of which are on and two of which are off. Every second, George randomly flips one of the switches, turning the state of the switch from on to off or from off to on. What is the expected number of seconds it takes until the state of every switch being on at the same time and the state of every switch being off at the same time have both occurred?

Proposed by Amudhan Gurumoorthy

Solution: $\boxed{\frac{88}{3}}$.

We will separate this problem into two parts, and approach each part with states. Let the switches' states be encoded by a 4-digit binary string with 1 representing "on" and 0 representing "off", and without loss of generality the initial state is 1100 (so the first two switches on, the last two off). Then we need to first get to one of $\{0000, 1111\}$, and then get to the other.

Part 1: Getting from 1100 to either one of $\{0000, 1111\}$. Consider the five different general states; 0 switches on, 1 switch on, 2 switches on, 3 switches on, and 4 switches on. We are initially at the 2-on state. When at the 2-on state, we have a $\frac{1}{2}$ chance of going to the 1-on state next and a $\frac{1}{2}$ chance of going to the 3-on state next. When at the 1-on state, we have a $\frac{1}{4}$ chance of going to the 0-on state next, and a $\frac{3}{4}$ chance of going to the 2-on state next. When at the 3-on state, we have a $\frac{1}{4}$ chance of going to the 4-on state next, and a $\frac{3}{4}$ chance of going to the 2-on state next. And when we are at either the 0-on or the 4-on states, we must go to the 1-on and the 3-on states next, respectively.

Let A_2 be the expected number of moves that it will take to reach either the 0-on state or the 4-on state, starting from the 2-on state. Let A_1 be the expected number of moves that it will take to reach either of the two goal states starting from the 1-on state. Note that A_1 is also the expected number of moves that it will take to reach either of the goal states starting from the 3-on state, by symmetry. Then we have the following equations:

$$A_2 = 1 + A_1$$

$$A_1 = 1 + \tfrac{3}{4}A_2 + \tfrac{1}{4} \cdot 0.$$

Solving, we get $A_1 = 7$ and $A_2 = 8$.

Part 2: Getting from one of $\{0000, 1111\}$ to the other. We will still use the five states (0-on, 1-on, 2-on, 3-on, 4-on). Without loss of generality, assume that we are currently at 1111 and wish to go to 0000. For $1 \leq i \leq 4$, let B_i be the expected number of moves that it will take to reach 0000 from the i-on state. Note that our goal is to find the value of B_4. Then we have the following equations:

$$B_4 = 1 + B_3$$

$$B_3 = 1 + \tfrac{3}{4}B_2 + \tfrac{1}{4}B_4$$

$$B_2 = 1 + \tfrac{1}{2}B_3 + \tfrac{1}{2}B_1$$

$$B_1 = 1 + \tfrac{3}{4}B_2 + \tfrac{1}{4} \cdot 0.$$

Solving, we get $B_4 = \frac{64}{3}$.

Thus, the total expected number of moves that it will take George to get from 1100 to one of $\{0000, 1111\}$ and then to the other is $A_2 + B_4 = 8 + \frac{64}{3} = \boxed{\frac{88}{3}}$.

Problem 2. How many ordered pairs of integers (a, b) with $1 \leq a, b \leq 1000$ are there such that $1000 \mid a^2 + b^2$?

Proposed by Vivian Loh

Solution: $\boxed{3400}$.

First, we will prove by induction that -1 is a quadratic residue $\pmod{5^k}$ for any $k \geq 1$. Note that -1 is a quadratic residue $\pmod 5$ since $2^2 \equiv -1 \pmod 5$. Assume that -1 is a quadratic residue $\pmod{5^{k-1}}$, so there is some residue $r \pmod{5^{k-1}}$ such that $r^2 \equiv -1 \pmod{5^{k-1}}$. Then we want some a such that $(5^{k-1}a+r)^2 \equiv -1 \pmod{5^k}$. Expanding, we just need $2ra \cdot 5^{k-1} \equiv -1-r^2 \pmod{5^k}$. Since $5^{k-1} \mid r^2+1$, $r^2+1 = b \cdot 5^{k-1}$ for some b. Then we need $2ra \equiv -b \pmod 5$, which clearly has a solution in a since $\gcd(r,5) = 1$. So, by induction, -1 is a quadratic residue $\pmod{125}$, so there exists some residue $r \pmod{125}$ such that $r^2 \equiv -1 \pmod{125}$.

We claim that the only residues $\pmod{125}$ whose squares are $-1 \pmod{125}$ are r and $-r$. This is because if $r^2 \equiv q^2 \pmod{125}$ then $125 \mid (r+q)(r-q)$, and these are clearly not both divisible by 5 (since $5 \nmid r$), so exactly one of them is divisible by 125, which means $q \equiv \pm r$.

Now, back to the problem:

Note that since perfect squares are only 0 or 1 $\pmod 4$, we must have a and b both even.

Case 1: $5 \mid a,b$. (Note that either 5 divides both or 5 divides neither, because of $\pmod 5$.) Then we just need $10 \mid \left(\frac{a}{10}\right)^2 + \left(\frac{b}{10}\right)^2$, where $1 \leq \frac{a}{10}, \frac{b}{10} \leq 100$. Thus, $(a,b) \equiv (0,0), (5,5)$, (either 1 or 9) and (either 3 or 7) in some order, or (either 2 or 8) and (either 4 or 6) in some order; a total of 18 possible ordered pairs $\pmod{10}$. Each residue class $\pmod{10}$ yields 10 possible values for a, so in total we have $18 \cdot 10 \cdot 10 = 1800$ ordered pairs in this case.

Case 2: $5 \nmid a,b$, $4 \mid a,b$. (Note that either $a,b \equiv 0 \pmod 4$ or $a,b \equiv 2 \pmod 4$, since $(2 \pmod 4)^2 \equiv 4 \pmod 8$ and $(0 \pmod 4)^2 \equiv 0 \pmod 8$ and we need the sum of the squares to be $0 \pmod 8$.) Let $a = 4a'$ and $b = 4b'$; then $1 \leq a', b' \leq 250$. For each residue $a' \pmod{125}$ that is not divisible by 5, there are 2 possible values for the residue of $b' \pmod{125}$ ($\pm a'r \pmod{125}$), and for each pair of residues $(a', b') \pmod{125}$, there are $2^2 = 4$ pairs (a,b) since $250 = 2 \cdot 125$. Since there are 100 residues $\pmod{125}$ that are not divisible by 5, the total number of ordered pairs in this case is $100 \cdot 2 \cdot 2^2 = 800$.

Case 3: $5 \nmid a,b$, $4 \nmid a,b$. Let $a = 2a'$ and $b = 2b'$; then $1 \leq a', b' \leq 500$ and a', b' are both odd. For each residue $a' \pmod{125}$ that's not divisible by 5, there are 2 possible residues for $b' \pmod{125}$ ($\pm a'r \pmod{125}$), and for each pair of residues $(a', b') \pmod{125}$, which also fixes $(a', b') \pmod{250}$, there are $2^2 = 4$ pairs (a,b) since $500 = 2 \cdot 250$. Since there are 100 residues $\pmod{125}$ that are not divisible by 5, the total number of ordered pairs in this case is $100 \cdot 2 \cdot 2^2 = 800$.

In total, there are $1800 + 800 + 800 = \boxed{3400}$ possible ordered pairs (a,b).

Problem 3. Points D, E, and F are randomly chosen on $\triangle ABC$, one on each side. What is the probability that the centroid of $\triangle DEF$ is closer to the centroid of $\triangle ABC$ than to any of the sides of $\triangle ABC$? *(Note: The centroid is the intersection of the medians in a triangle.)*

Proposed by Vivian Loh

Solution: $\boxed{\frac{5}{8}}$.

We will use barycentric coordinates. The vertices of the triangle are $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, and $D = (a, 1-a, 0)$, $E = (0, b, -b)$, $F = (1-c, 0, c)$ where a,b,c are randomly chosen numbers between 0 and 1. The centroid of $\triangle DEF$ is $\frac{D+E+F}{3} = \left(\frac{a+1-c}{3}, \frac{b+1-a}{3}, \frac{c+1-b}{3}\right)$, and this point is closer to the centroid $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ if and only if each of the coordinates is $> \frac{1}{6}$. So we need

$$\frac{a+1-c}{3} > \frac{1}{6}, \frac{b+1-a}{3} > \frac{1}{6}, \frac{c+1-b}{3} > \frac{1}{6}$$

which is equivalent to

$$c-a < \frac{1}{2}, a-b < \frac{1}{2}, b-c < \frac{1}{2}.$$

Note that since $0 < a,b,c < 1$, at most one of these conditions can be violated at once, so the chance of this holding is just 1 minus 3 times the probability of $c - a > \frac{1}{2}$. Thus, by geometric probability, the answer is $1 - 3 \cdot \frac{1}{8} = \boxed{\frac{5}{8}}$.

Problem 4. Complex numbers a, b, c are such that

$$\left|-\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right| = \left|\frac{1}{a}-\frac{1}{b}+\frac{1}{c}\right| = \left|\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right| = 1.$$

If $|a| = 2025$, then the minimum value of $|b-c|$ is equal to d. What is d^2? *(Note: $|z|$ denotes the magnitude of a complex number, or its distance from the origin in the complex plane.)*

Proposed by Vivian Loh

Solution: $\boxed{\frac{1013}{253}}$.

Consider the system of equations

$$\frac{2}{a} = e+f, \frac{2}{b} = d+f, \frac{2}{c} = d+e.$$

Solving them, we get

$$d = \tfrac{1}{b}+\tfrac{1}{c}-\tfrac{1}{a}, e = \tfrac{1}{a}+\tfrac{1}{c}-\tfrac{1}{b}, f = \tfrac{1}{a}+\tfrac{1}{b}-\tfrac{1}{c},$$

so the condition given in the problem is equivalent to d, e, f all lying on the unit circle. Note that the triangle formed by $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ in the complex plane is the medial triangle of the one formed by d, e, f, so the segment between $\frac{1}{b}$ and $\frac{1}{c}$ is half the length of the segment between e and f, and also parallel to it. Furthermore, the segment between 0 and $\frac{1}{a}$ is perpendicular to the one through e and f, and we know that $\left|\frac{1}{a}\right| = \frac{1}{2025}$, so by the Pythagorean Theorem, the segment between $\frac{1}{b}$ and $\frac{1}{c}$ has length $\sqrt{1-\frac{1}{2025^2}}$, and so this is $\left|\frac{1}{b}-\frac{1}{c}\right|$.

Now by the Inversion Distance Formula, or just by noting that the triangle formed by 0, $\frac{1}{b}$, $\frac{1}{c}$ is similar to the one formed by 0, c, b, we have that the distance between b and c, which is $|b-c|$, is

$$\frac{\left|\frac{1}{b}-\frac{1}{c}\right|}{\left|\frac{1}{b}\right| \cdot \left|\frac{1}{c}\right|} = \frac{\sqrt{1-\frac{1}{2025^2}}}{\left|\frac{1}{b}\right| \cdot \left|\frac{1}{c}\right|}.$$

So now we just want to maximize $\left|\frac{1}{b}\right| \cdot \left|\frac{1}{c}\right|$. Note that $\frac{1}{c}$ and $\frac{1}{b}$ are corresponding points under the translation sending the circle through 0, $\frac{1}{a}$, e to the circle through 0, $\frac{1}{a}$, f. Let points 0, $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, d, e, f be denoted O, A^*, B^*, C^*, D, E, F respectively. Then $\angle B^*OC^*$ is equal to either $\angle EOA^*$ or $180^\circ - \angle EOA^*$ depending on whether or not O is inside triangle $\triangle A^*B^*C^*$, but $\sin\angle B^*OC^*$ is always the same, so by the Sine Area Formula, $OB^* \cdot OC^*$ is maximized when $[OB^*C^*]$ or equivalently the distance from O to B^*C^* is maximized, which occurs when B^* and C^* are the midpoints of arcs $\overset{\frown}{FA^*}$ and $\overset{\frown}{EA^*}$ of (OFA^*) and (OEA^*) not including O.

By the Pythagorean Theorem, the maximum value of $OB^* \cdot OC^*$ is $\frac{1}{4}(B^*A^{*\prime})^2$ where $A^{*\prime}$ is the point on the unit circle diametrically opposite A^*, and this is

$$\frac{1}{4}\left(\left(\frac{2024}{2025}\right)^2 + \left(\frac{\sqrt{2025^2-1}}{2025}\right)^2\right) = \frac{1}{4} \cdot \frac{2024 \cdot 2025 \cdot 2}{2025^2} = \frac{1012}{2025}$$

so the minimum value of $|b-c|$ is

$$\frac{\sqrt{1-\frac{1}{2025^2}}}{\frac{1012}{2025}} = \frac{\frac{\sqrt{2024 \cdot 2026}}{2025}}{\frac{1012}{2025}} = \frac{\sqrt{2024 \cdot 2026}}{1012} = d$$

$$\implies d^2 = \frac{2024 \cdot 2026}{1012^2} = \frac{2 \cdot 2026}{1012} = \boxed{\frac{1013}{253}}.$$