CHMMC 2020-2021

Individual Round

1. A right triangle ABC is inscribed in the circular base of a cone. If two of the side lengths of ABC are 3 and 4, and the distance from the vertex of the cone to any point on the circumference of the base is 3, then the minimum possible volume of the cone can be written as $\frac{m\pi\sqrt{n}}{p}$, where m, n, and p are positive integers, m and p are relatively prime, and n is squarefree. Find $m+n+p$.

2. Caltech's 900 students are evenly spaced along the circumference of a circle. How many equilateral triangles can be formed with at least two Caltech students as vertices?

3. A *Beaver-number* is a positive 5 digit integer whose digit sum is divisible by 17. Call a pair of *Beaver-numbers* differing by exactly 1 a *Beaver-pair*. The smaller number in a *Beaver-pair* is called an *MIT Beaver*, while the larger number is called a *CIT Beaver*. Find the positive difference between the largest and smallest *CIT Beavers* (over all *Beaver-pairs*).

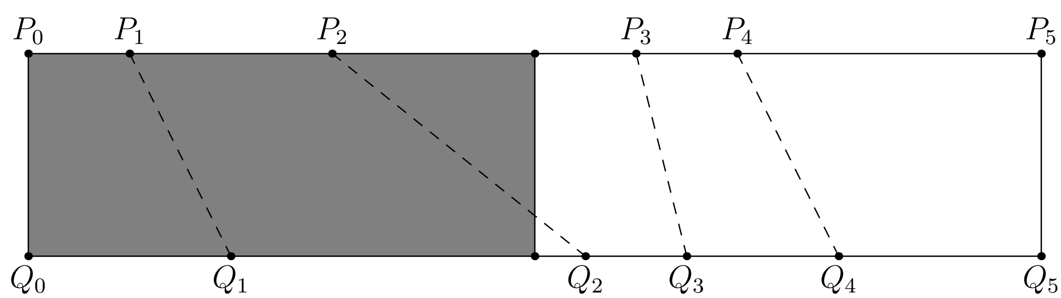
4. Let $P(x) = x^3 - 6x^2 - 5x + 4$. Suppose that y and z are real numbers such that

$$zP(y) = P(y-n) + P(y+n)$$

for all reals n. Evaluate $P(y)$.

5. Let S be the sum of all positive integers n such that $\frac{3}{5}$ of the positive divisors of n are multiples of 6 and n has no prime divisors greater than 3. Compute $\frac{S}{36}$.

6. Let $P_0P_5Q_5Q_0$ be a rectangular chocolate bar, one half dark chocolate and one half white chocolate, as shown in the diagram below. We randomly select 4 points on the segment P_0P_5, and immediately after selecting those points, we label those 4 selected points P_1, P_2, P_3, P_4 from left to right. Similarly, we randomly select 4 points on the segment Q_0Q_5, and immediately after selecting those points, we label those 4 points Q_1, Q_2, Q_3, Q_4 from left to right. The segments $P_1Q_1, P_2Q_2, P_3Q_3, P_4Q_4$ divide the rectangular chocolate bar into 5 smaller trapezoidal pieces of chocolate. The probability that exactly 3 pieces of chocolate contain both dark and white chocolate can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



7. Given 10 points on a plane such that no three are collinear, we connect each pair of points with a segment and color each segment either red or blue. Assume that there exists some point A among the 10 points such that:

(1) There is an odd number of red segments connected to A

(2) The number of red segments connected to each of the other points are all different

Find the number of red triangles (i.e, a triangle whose three sides are all red segments) on the plane.

8. Define

$$S = \tan^{-1}(2020) + \sum_{j=0}^{2020} \tan^{-1}(j^2 - j + 1).$$

Then S can be written as $\frac{m\pi}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

9. For a positive integer m, let $\varphi(m)$ be the number of positive integers $k \leq m$ such that k and m are relatively prime, and let $\sigma(m)$ be the sum of the positive divisors of m. Find the sum of all even positive integers n such that

$$\frac{n^5\sigma(n) - 2}{\varphi(n)}$$

is an integer.

10. A research facility has 60 rooms, numbered $1, 2, \ldots 60$, arranged in a circle. The entrance is in room 1 and the exit is in room 60, and there are no other ways in and out of the facility. Each room, except for room 60, has a teleporter equipped with an integer instruction $1 \leq i < 60$ such that it teleports a passenger exactly i rooms clockwise.

On Monday, a researcher generates a random permutation of $1, 2, \ldots, 60$ such that 1 is the first integer in the permutation and 60 is the last. Then, she configures the teleporters in the facility such that the rooms will be visited in the order of the permutation.

On Tuesday, however, a cyber criminal hacks into a randomly chosen teleporter, and he reconfigures its instruction by choosing a random integer $1 \leq j' < 60$ such that the hacked teleporter now teleports a passenger exactly j' rooms clockwise (note that it is possible, albeit unlikely, that the hacked teleporter's instruction remains unchanged from Monday). This is a problem, since it is possible for the researcher, if she were to enter the facility, to be trapped in an endless "cycle" of rooms.

The probability that the researcher will be unable to exit the facility after entering in room 1 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

11. Let $n \geq 3$ be a positive integer. Suppose that Γ is a unit circle passing through a point A. A regular 3-gon, regular 4-gon, $\ldots$, regular n-gon are all inscribed inside Γ such that A is a common vertex of all these regular polygons. Let Q be a point on Γ such that Q is a vertex of the regular n-gon, but Q is not a vertex of any of the other regular polygons. Let $\mathcal{S}_n$ be the set of all such points Q. Find the number of integers $3 \leq n \leq 100$ such that

$$\prod_{Q \in \mathcal{S}_n} |AQ| \leq 2.$$

12. Let Ω_1 and Ω_2 be two circles intersecting at distinct points P and Q. The line tangent to Ω_1 at P passes through Ω_2 at a second point A, and the line tangent to Ω_2 at P passes through Ω_1 at a second point B. Ray AQ intersects Ω_1 at a second point C, and ray BQ intersects Ω_2 at a second point D. Suppose that $\angle CPD > \angle APB$ (measuring both angles as the non-reflex angle) and that

$$\frac{\text{Area}(CPD)}{PA \cdot PB} = \frac{1}{4}.$$

Find the sum of all possible measures of $\angle APB$ in degrees.

13. Let a, b, c, d be reals such that $a \geq b \geq c \geq d$ and

$$\begin{aligned}&(a-b)^3 + (b-c)^3 + (c-d)^3 - 2(d-a)^3\\&- 12(a-b)^2 - 12(b-c)^2 - 12(c-d)^2 + 12(d-a)^2\\&- 2020(a-b)(b-c)(c-d)(d-a) = 1536.\end{aligned}$$

Find the minimum possible value of $d - a$.

14. Let a be a positive real number. Collinear points Z_1, Z_2, Z_3, Z_4 (in that order) are plotted on the (x, y) Cartesian plane. Suppose that the graph of the equation

$$x^2 + (y+a)^2 + x^2 + (y-a)^2 = 4a^2 + \sqrt{(x^2 + (y+a)^2)(x^2 + (y-a)^2)}$$

passes through points Z_1 and Z_4, and the graph of the equation

$$x^2 + (y+a)^2 + x^2 + (y-a)^2 = 4a^2 - \sqrt{(x^2 + (y+a)^2)(x^2 + (y-a)^2)}$$

passes through points Z_2 and Z_3. If $Z_1Z_2 = 5$, $Z_2Z_3 = 1$, and $Z_3Z_4 = 3$, then a^2 can be written as $\frac{m+n\sqrt{p}}{q}$, where m, n, p, and q are positive integers, m, n, and q are relatively prime, and p is squarefree. Find $m + n + p + q$.

15. For an integer $n \geq 2$, let G_n be an $n \times n$ grid of unit cells. A subset of cells $H \subseteq G_n$ is considered *quasi-complete* if and only if each row of G_n has at least one cell in H and each column of G_n has at least one cell in H. A subset of cells $K \subseteq G_n$ is considered *quasi-perfect* if and only if there is a proper subset $L \subset K$ such that $|L| = n$ and no two elements in L are in the same row or column. Let $\vartheta(n)$ be the smallest positive integer such that every quasi-complete subset $H \subseteq G_n$ with $|H| \geq \vartheta(n)$ is also quasi-perfect. Moreover, let $\varrho(n)$ be the number of quasi-complete subsets $H \subseteq G_n$ with $|H| = \vartheta(n) - 1$ such that H is not quasi-perfect. Compute $\vartheta(20) + \varrho(20)$.