Ramsey Theory

Caltech Math Meet 2025

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Motivating problem

Example

What is the smallest number of people needed in a room to guarantee that three people are mutual friends or mutual strangers?

Definition of a graph

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A *graph* $G = (V, E)$ is a collection of *vertices* and *edges* or pairs of vertices.

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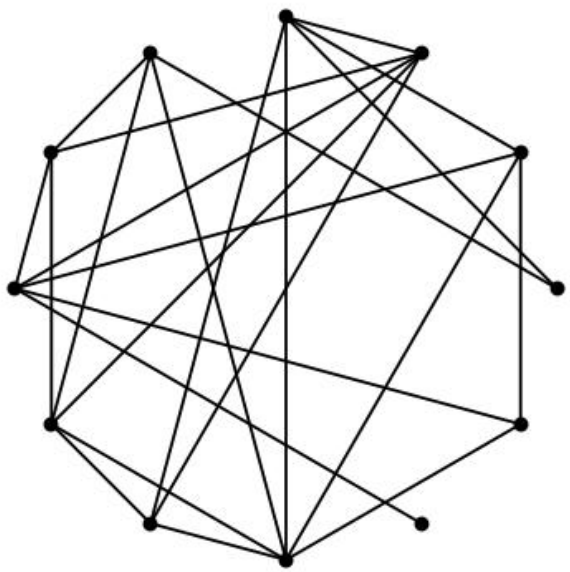

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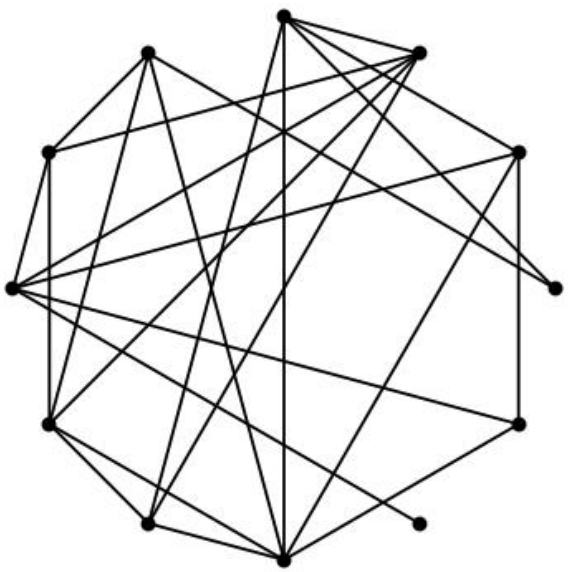

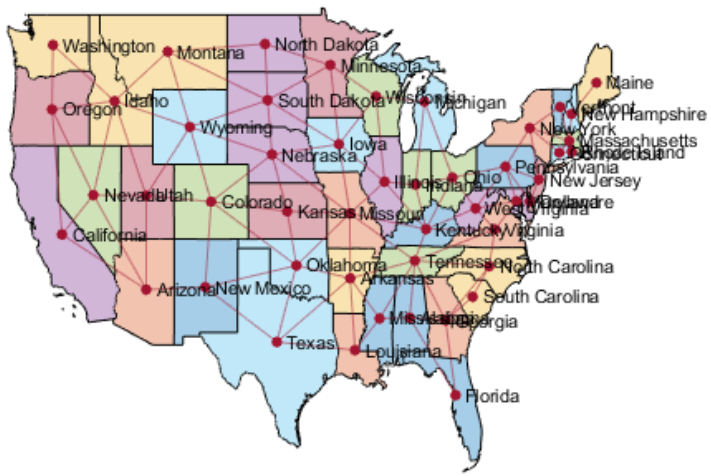

Complete graph

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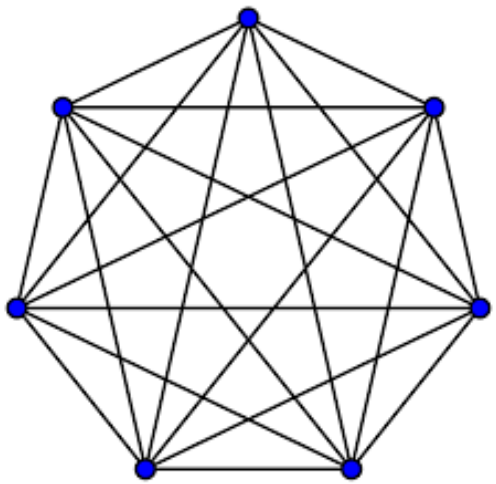
The *complete graph* on n vertices, denoted K_n, has an edge between any pair of vertices.



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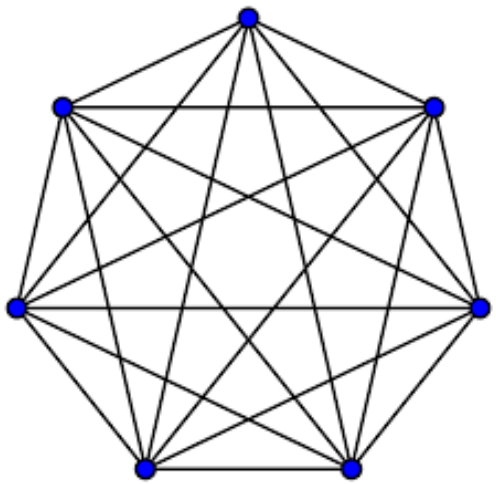

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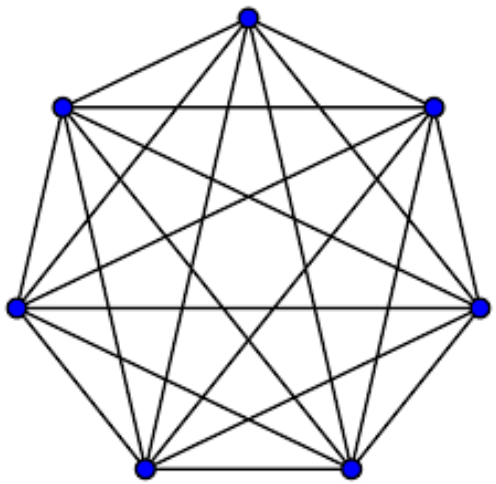

How many edges does the complete graph on n vertices have?

$$|E(K_n)| = \frac{n(n-1)}{2}$$

Edge-coloring of complete graphs

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If we represent each person as a vertex in a complete graph, we can color an edge blue to denote friends and red to denote strangers.

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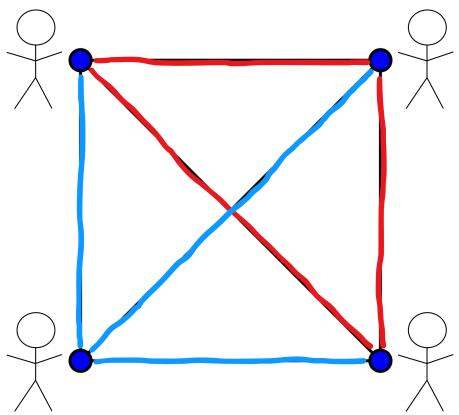
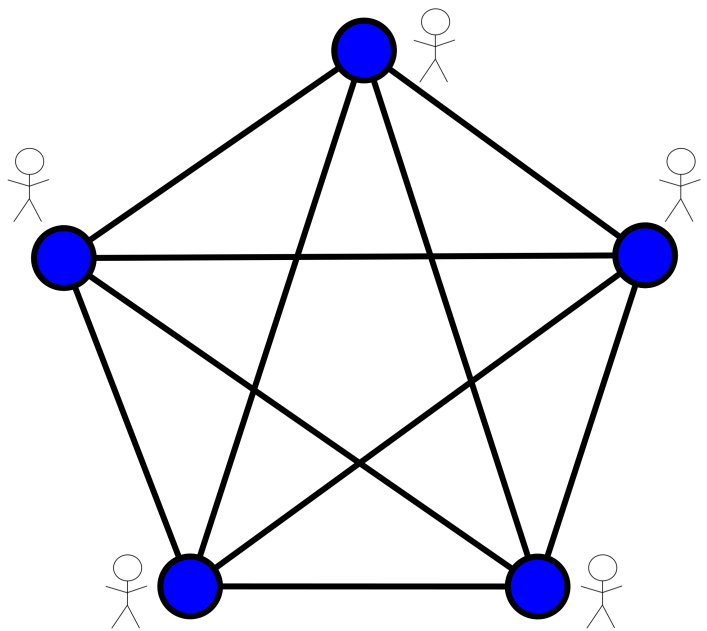


Top left, top right, and bottom right are mutual strangers.

Motivating problem

Example

What is the smallest n for which any 2-coloring of the edges of K_n yields a monochromatic triangle?



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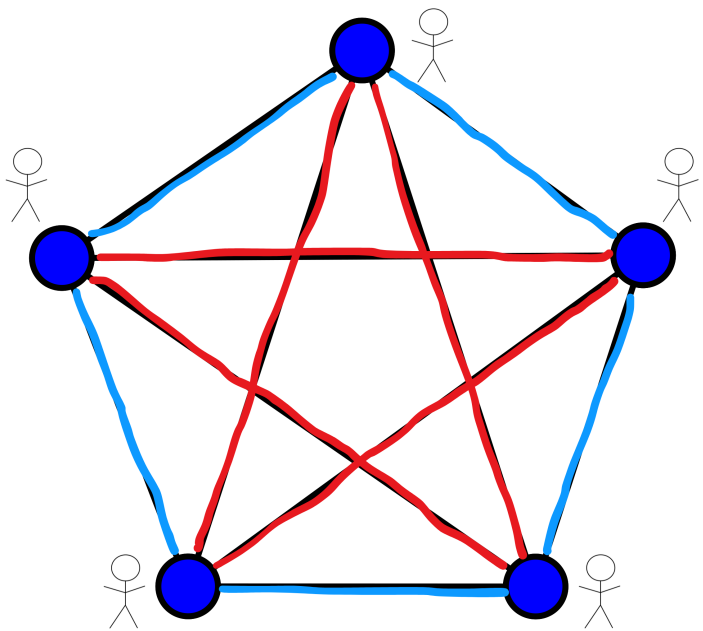

Pigeonhole principle

On the previous slide, we argued that the answer to our motivating problem is more than 5. We now want to prove that any 2-coloring of the edges of K_6 will have either a blue triangle or red triangle.

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Proposition (Pigeonhole principle)

If n items are put into m containers with $n > m$, then at least one container must contain more than one item.

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Lemma

Any red/blue-coloring of the edges of K_6 will either have a blue triangle or a red triangle.

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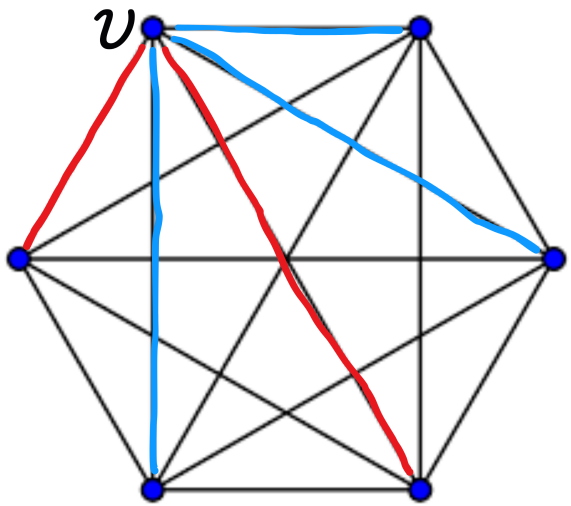
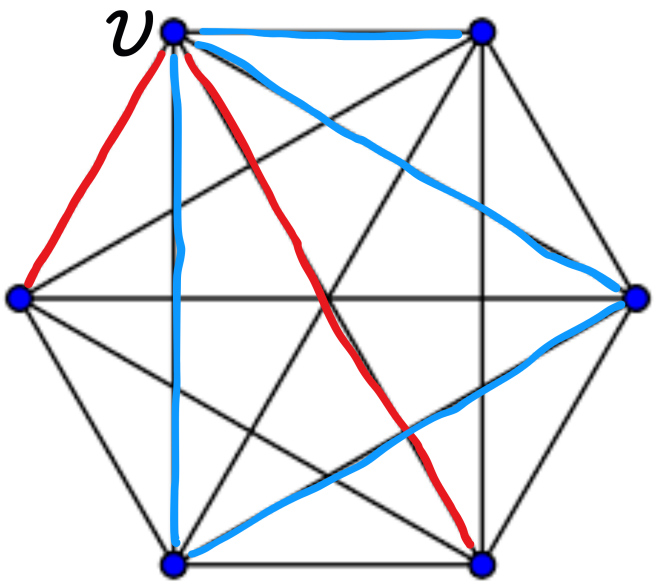
Any red/blue-coloring of the edges of K_6 will either have a blue triangle or a red triangle.

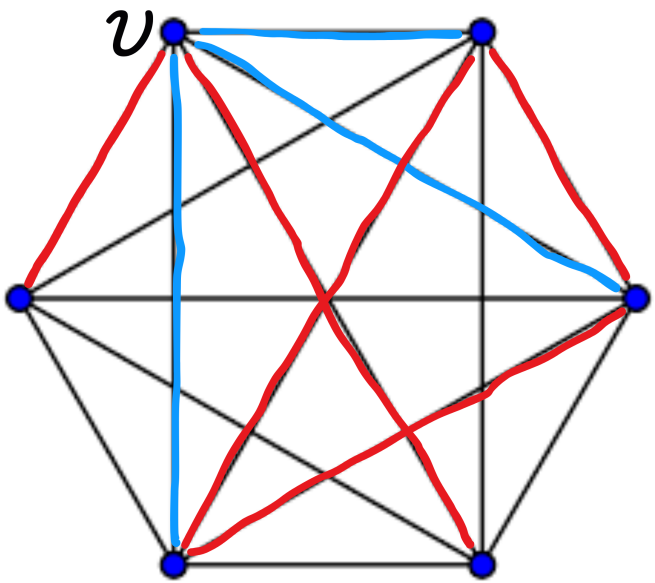
Think of edges out of v as pigeons and colors as pigeonholes. By pigeonhole principle, v has at least three red neighbors or at least three blue neighbors (pictured).



Motivating problem

Vertices connected to v by blue edges form the *blue neighborhood*.





Any blue edge between vertices in the blue neighborhood results in a blue triangle (left picture). If all the edges between vertices in the blue neighborhood are red, we have a red triangle (right picture).

Ramsey numbers

Proposition

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The *Ramsey number* $R(s, t)$ is the smallest n for which a 2-coloring of the edges of K_n must contain a blue K_s or a red K_t.

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The motivating problem was to find $R(3, 3)$ since a triangle is a complete graph on three vertices.

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$$R(2, t) = t$$

Color each edge of K_{t-1} red. The graph will contain neither a blue K_2 nor a red K_t.

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$$R(2, t) = t$$

Color each edge of K_{t-1} red. The graph will contain neither a blue K_2 nor a red K_t.

If we color any edge blue in K_t, we are guaranteed a blue complete graph on two vertices. If we color all edges of K_t red, then we have a red copy of K_t.

Ramsey's Theorem and Erdős–Szekeres inequality

Theorem (Ramsey's Theorem)

There exists a least positive integer $R(s,t)$ for which every 2-coloring of the edges of the complete graph on $R(s,t)$ vertices contains a blue K_s or a red K_t.

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Theorem (Erdős–Szekeres inequality)

For integers $s, t \geq 2$, $R(s+1, t+1) \leq R(s, t+1) + R(s+1, t)$.

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Theorem (Erdős–Szekeres inequality)

For integers $s,t \geq 2$, $R(s+1,t+1) \leq R(s,t+1) + R(s+1,t)$.

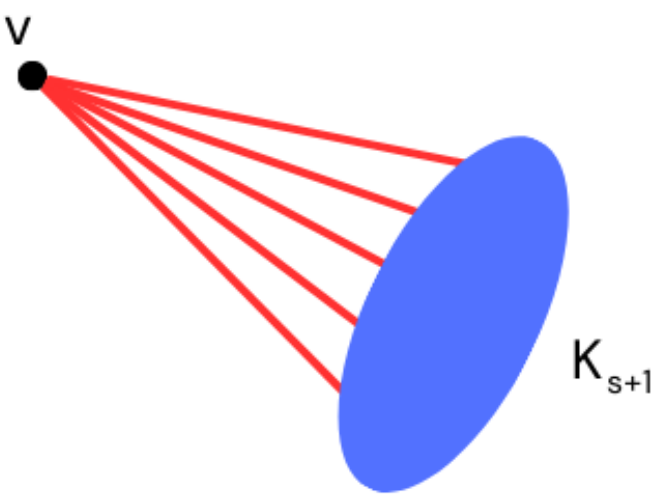
Corollary (Erdős–Szekeres, 1935)

For all natural numbers $s,t \geq 1$, $R(s+1,t+1) \leq \binom{s+t}{s}$.

Erdős–Szekeres inequality

Proof.

Color each edge of $K_{R(s,t+1)+R(s+1,t)}$ red or blue. Choose vertex v. If the red neighborhood of v has at least $R(s+1,t)$ vertices, the neighborhood contains a blue K_{s+1} or a red K_t. Adding v to the red K_t produces a red K_{t+1}. Make a similar argument if the blue neighborhood of v has at least $R(s,t+1)$ vertices. □



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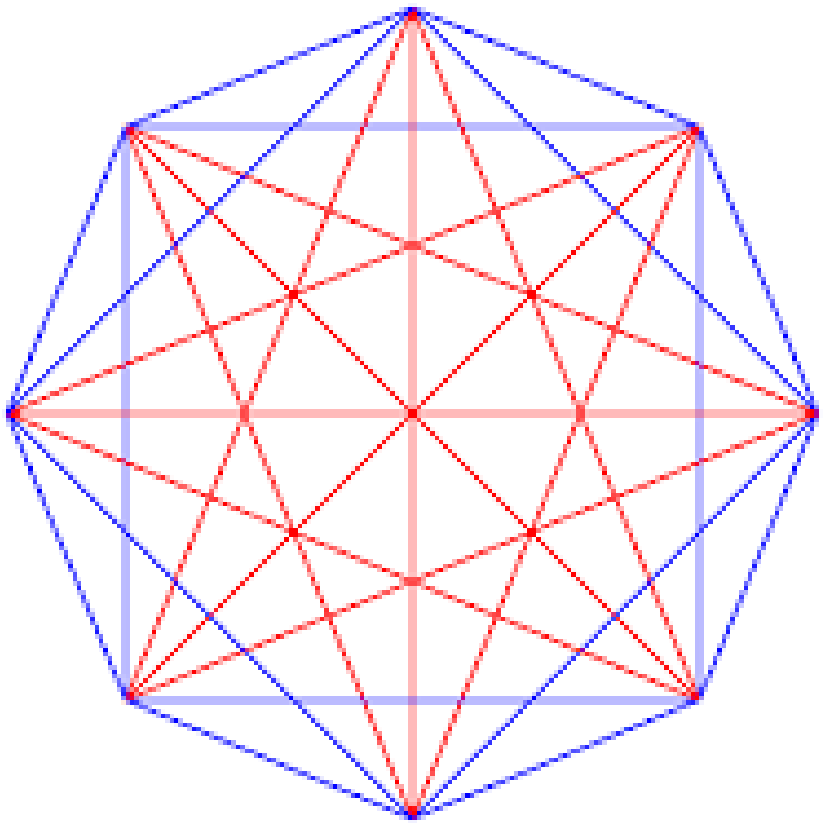
Erdős–Szekeres inequality



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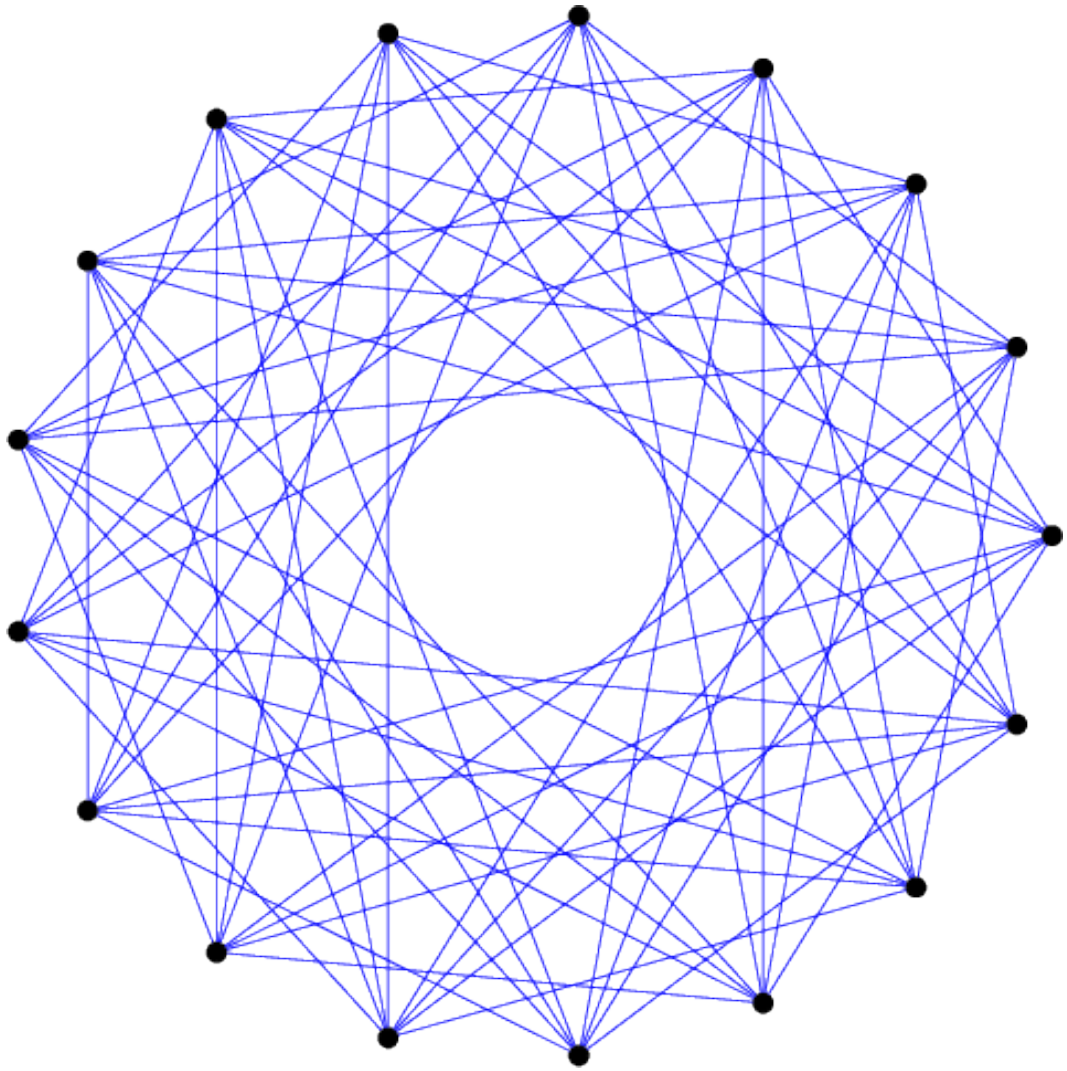
Erdős–Szekeres inequality



Why are Ramsey numbers difficult to compute?

"Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5,5)$ or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value.

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But suppose, instead, that they ask for $R(6,6)$. In that case, he believes, we should attempt to destroy the aliens."

–Joel Spencer

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There are $\frac{n(n-1)}{2}$ edges. Each edge can be red or blue, so there are $2^{\frac{n(n-1)}{2}}$ ways to color the edges of K_n. Many of these edge-colorings are the same, but even determining repeats is an unsolved problem.

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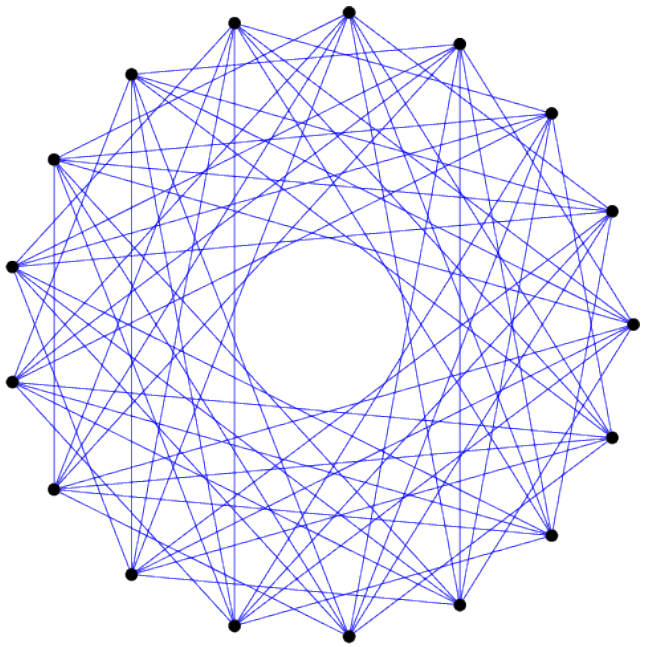
According to some estimates, there are roughly 10^{80} atoms in the observable universe. The complete graph on 26 vertices already has more red/blue-colorings of its edges than there are particles in the observable universe.

Why are Ramsey numbers difficult to compute?

The blue graph we used to find a lower bound for $R(4,4) = 18$ is a Paley graph. Treat each vertex like a number modulo 17. Connect two vertices a and b if $a - b$ is a square modulo 17.



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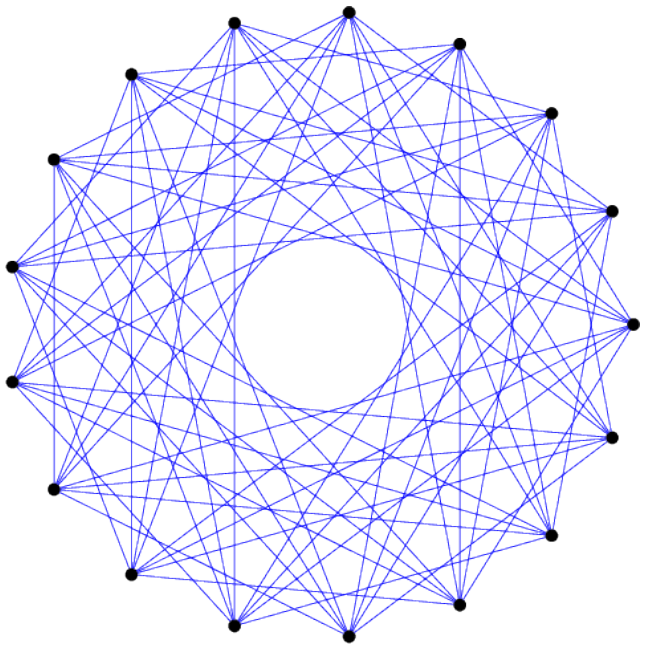

Of the over 2.46×10^{26} edge-colorings of K_{17}, the Paley graph above is the *only* one that works as a lower bound for $R(4,4)$.

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$$2^{1-\binom{t}{2}}\binom{n}{t} < 2^{1-\binom{t}{2}}\frac{n^t}{t!} \leq 2^{1-\binom{t}{2}}\left(\frac{t2^{(t-1)/2}}{e}\right)^t \frac{1}{2}\left(\frac{e}{t}\right)^t = 1$$

The expectation is smaller than 1, so there must be a coloring of K_n that does not contain a monochromatic K_t. □

Known Ramsey numbers

r \ s	1	2	3	4	5	6	7	8	9	10
1	**1**	1	1	1	1	1	1	1	1	1
2		**2**	3	4	5	6	7	8	9	10
3			**6**	9	14	18	23	28	36	40–41[14]
4				**18**	25[9]	36–40	49–58	59[15]–79	73–105	92–135
5					**43–46**[11]	59[16]–85	80–133	101–193	133–282	149[15]–381
6						**102–160**	115[15]–270	134[15]–423	183–651	204–944
7							**205–492**	219–832	252–1368	292–2119
8								**282–1518**	329–2662	343–4402
9									**565–4956**	581–8675
10										**798–16064**

Source: Wikipedia, "Ramsey's theorem"

Recent progress

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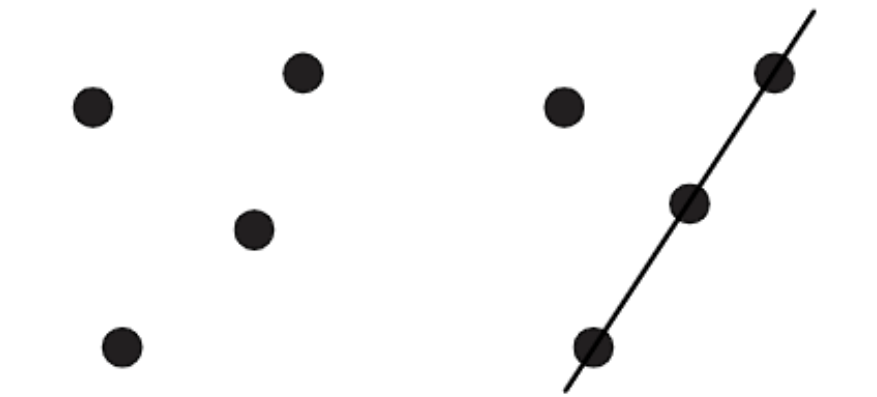
The bound $R(5,5) \leq 46$ is from 2024.

Lower bounds for $R(6,6)$ and $R(8,8)$ have not been improved since 1965 and 1972, respectively.

General position

Definition

A set of points in the plane is in *general position* if no three points lie on the same line.



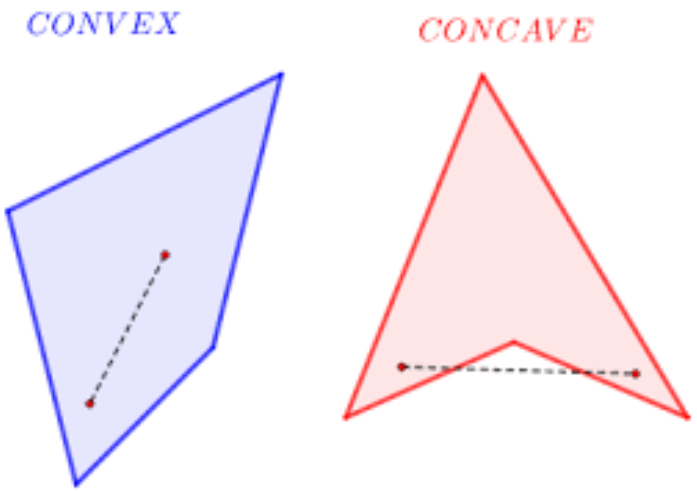
points in general position

*points **not** in general position*

Convex polygon

Definition

A polygon is *convex* if the line segment between any two points in the polygon is contained in the polygon.



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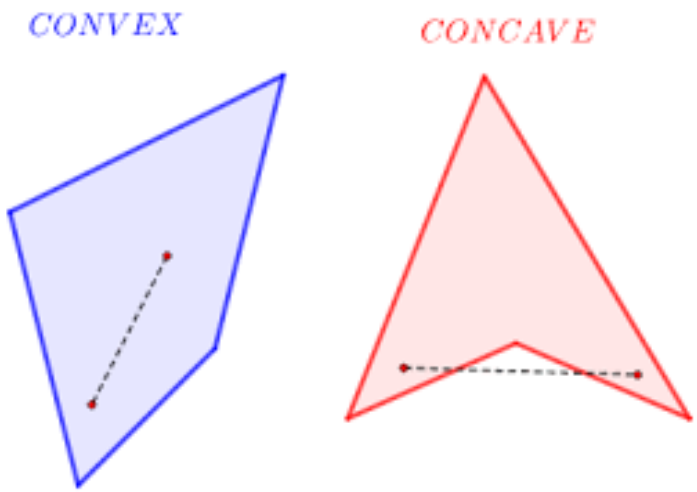

Example

If we place three points in general position in the plane, then the points are the vertices of a convex triangle.

Convex quadrilateral

Example

If we place four points in general position in the plane, are the points necessarily the vertices of a convex quadrilateral?

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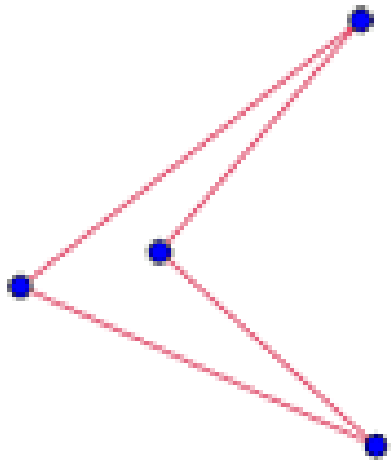

Convex quadrilateral

Example

If we place five points in general position in the plane, do four of the points necessarily form the vertices of a convex quadrilateral?

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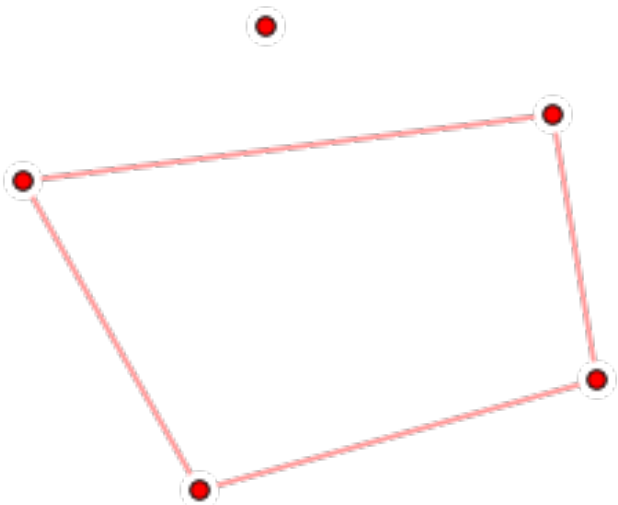
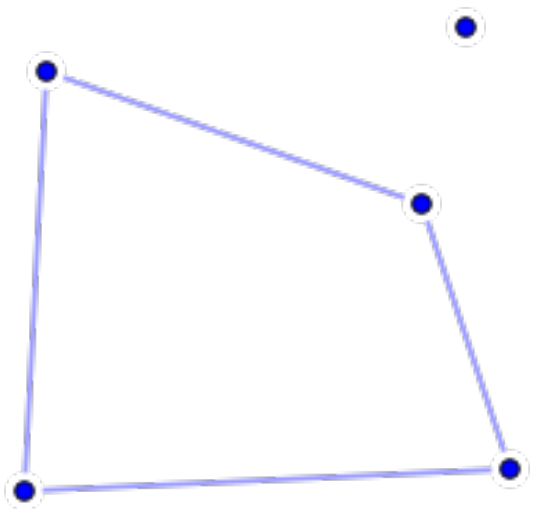
Example

If we place five points in general position in the plane, do four of the points necessarily form the vertices of a convex quadrilateral?





Happy ending problem

Theorem (Klein)

Any set of five points in the plane in general position has a subset of four points that form the vertices of a convex quadrilateral.

Happy ending problem

Proof.

If the five points form the vertices of a convex pentagon, we are done. Likewise, if four of the points form the vertices of a convex quadrilateral with the fifth point in the middle, we are done.

Happy ending problem

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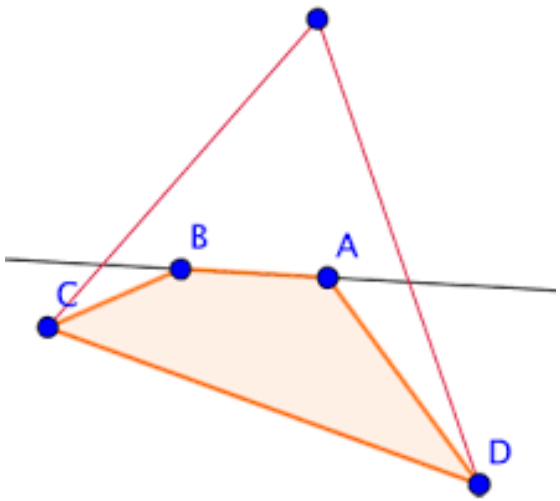
If the five points form the vertices of a convex pentagon, we are done. Likewise, if four of the points form the vertices of a convex quadrilateral with the fifth point in the middle, we are done. The only case to consider is when two points A and B sit inside the triangle formed by the other three points. □



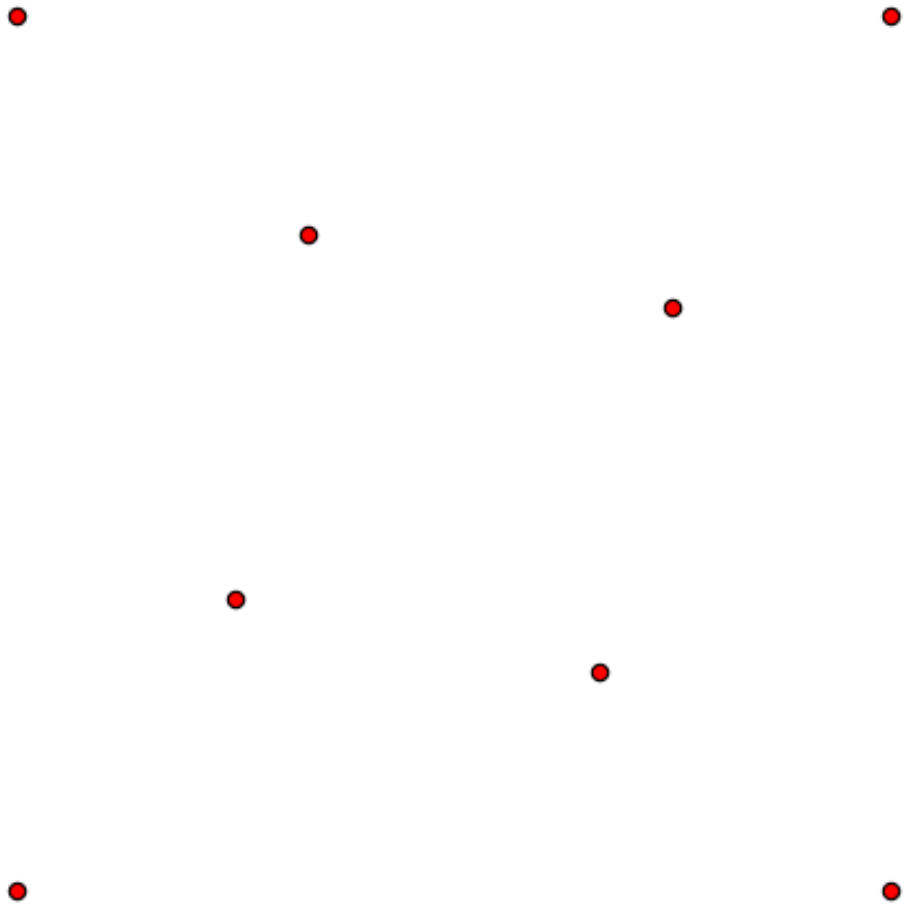
Happy ending problem

Example

Is there a smallest M for which any set of M points in the plane in general position has n points that form vertices of a convex n-gon?

n-gon	M
3	3
4	5
5	
6	

Happy ending problem



Happy ending problem

The previous image proves $M > 8$.

We can also argue that any 9 points in the plane in general position must contain the vertices of a convex pentagon.

n-gon	M
3	3
4	5
5	9
6	

Happy ending problem

Happy ending problem

n-gon	M
3	3
4	5
5	9
6	17

Happy ending problem

n-gon	M
3	3
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Conjecture (Erdős)

The smallest number of points in the plane in general position that guarantees n points form the vertices of a convex n-gon is $2^{n-2}+1$.

Happy ending problem

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Conjecture (Erdős)

The smallest number of points in the plane in general position that guarantees n points form the vertices of a convex n-gon is $2^{n-2}+1$.

Proposition (Erdős–Szekeres)

There is a way to arrange 2^{n-2} points in the plane so that no n of the points form the vertices of a convex n-gon.

Happy ending problem

Theorem (Happy ending problem)

For any natural number n, there exists M such that any set of M points in general position in the plane contains n points that form the vertices of a convex n-gon.

Their original proof relies on hypergraph Ramsey numbers.

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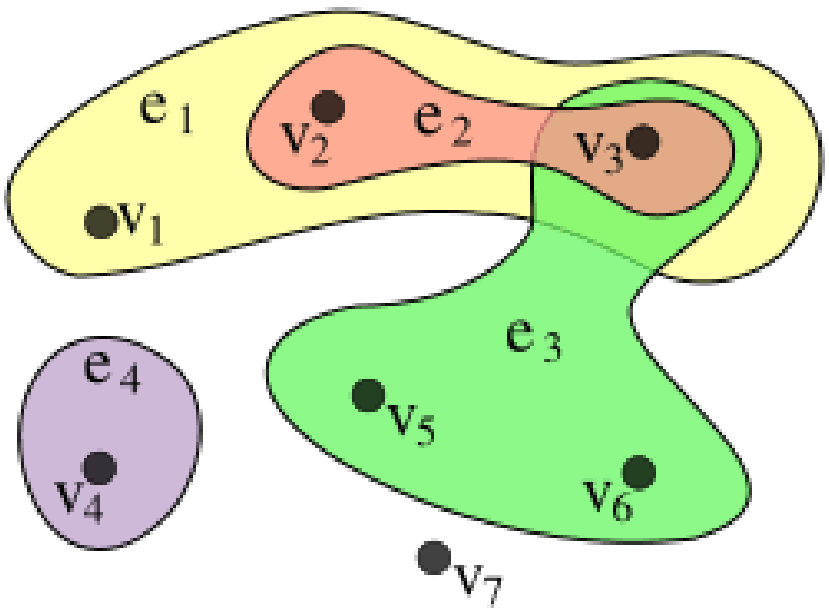
The result is known as the "Happy ending problem" because it led to the marriage of Klein and Szekeres.

Definition of a hypergraph

Definition

A *hypergraph* $G = (V, E)$ is a collection of *vertices* and *hyperedges* or subsets of vertices of any size.



Complete k-uniform hypergraphs

Definition

A hypergraph is *k-uniform* if each hyperedge has size k.

Example

A graph is a 2-uniform hypergraph.

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Example

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Definition

A complete k-uniform hypergraph on n vertices, denoted $K_n^{(k)}$, has every subset of k vertices as a hyperedge.

Hypergraph Ramsey numbers

Theorem (Hypergraph Ramsey's Theorem)

For any natural numbers s, t and k at least two, there exists an N such that every red/blue-coloring of the edge set of $K_N^{(k)}$ contains either a blue copy of $K_s^{(k)}$ or a red copy of $K_t^{(k)}$.

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Definition

The smallest N for which Ramsey's Theorem holds is the *hypergraph Ramsey number* $R_k(s, t)$.

Proof of the happy ending problem

Theorem (Happy ending problem)

For any natural number n, there exists M such that any set of M points in general position in the plane contains n points that form the vertices of a convex n-gon.

Lemma

If every subset of 4 points forms a convex quadrilateral, then n points in general position form a convex n-gon.

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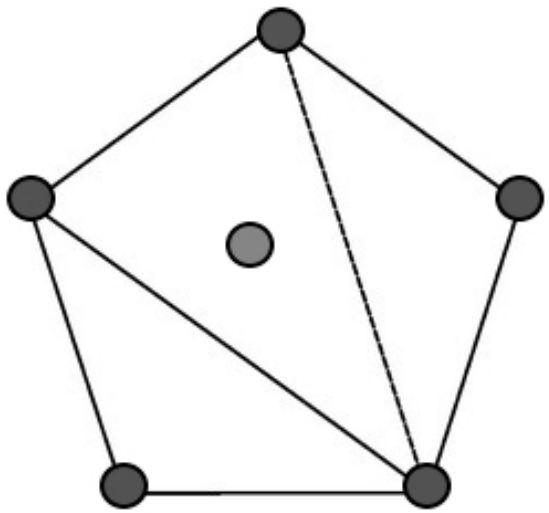
If every subset of 4 points forms a convex quadrilateral, then n points in general position form a convex n-gon.

Proof.

Assume that one of the n points x lies in the interior of a convex m-gon. Triangulate the m-gon. Since the points are in general position, x is contained in a convex triangle. □



Theorem (Happy ending problem)

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The Ramsey number $M = R_4(5, n)$ exists by Ramsey's Theorem.

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The Ramsey number $M = R_4(5, n)$ exists by Ramsey's Theorem. Let X be a set of M points in general position. Color a 4-tuple of points from X blue if one of the points is contained in the triangle formed by the other three. Color a 4-tuple red otherwise.

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Proof of the happy ending problem.

The Ramsey number $M = R_4(5, n)$ exists by Ramsey's Theorem. Let X be a set of M points in general position. Color a 4-tuple of points from X blue if one of the points is contained in the triangle formed by the other three. Color a 4-tuple red otherwise. By Klein's result, among any 5 points there are necessarily 4 points in convex position, so the coloring cannot contain a blue $K_5^{(4)}$. Thus, the coloring must contain a red $K_n^{(4)}$, which forms a convex n-gon by the Lemma. □

Schur's Theorem

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When the natural numbers are ℓ-colored, there is a solution to $x + y = z$ in one color class.

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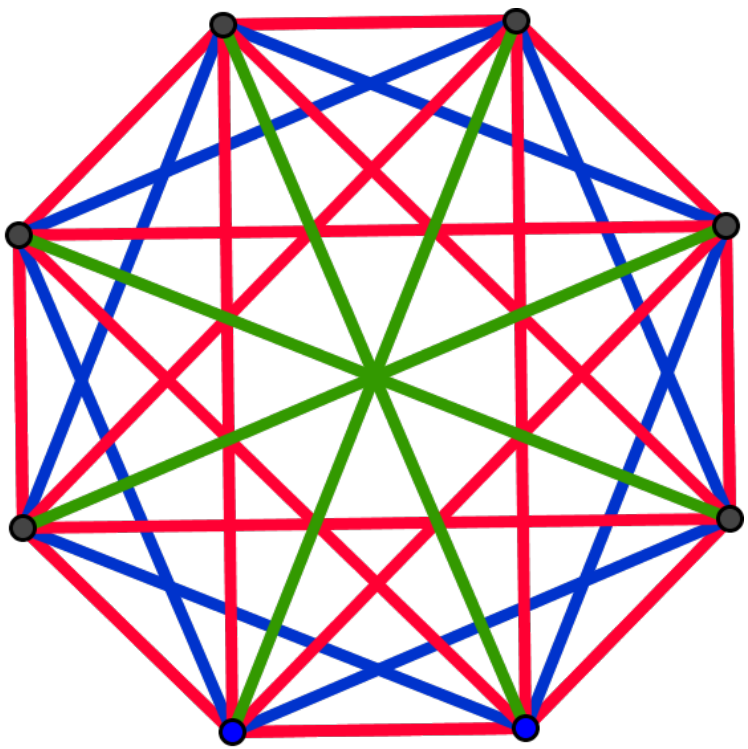
Example

Color a natural number blue if it is congruent to 0 modulo 3, red if it is congruent to 1 modulo 3, or green if it is congruent to 2 modulo 3. Then $3 + 6 = 9$ within the blue color class. However, there is no triple that works in either of the other two colors.

Multicolored Ramsey number

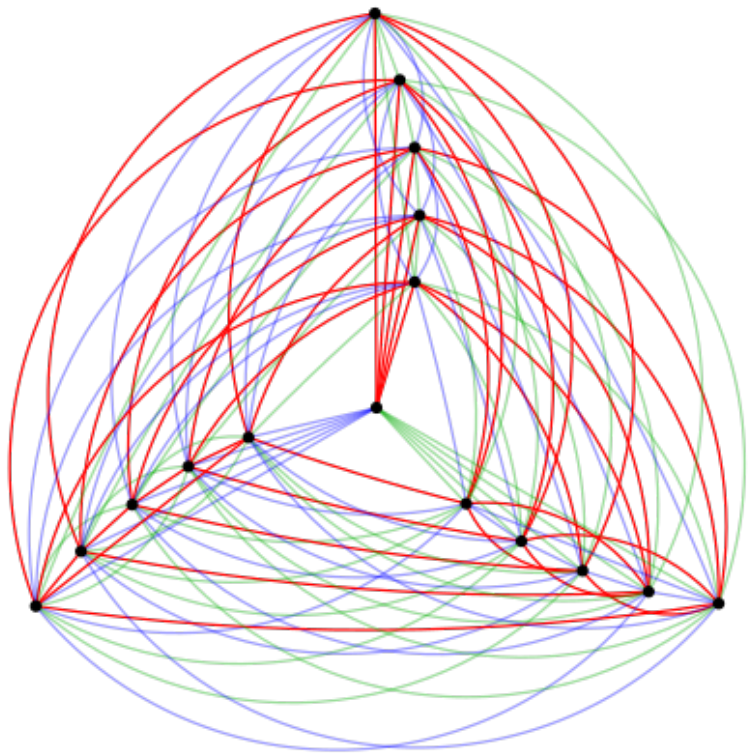
Definition

The *multicolor Ramsey number* $R(s_1, \ldots, s_\ell)$ is the smallest N for which K_N must contain some K_{s_i} of color i when the edges of K_N are ℓ-colored.



Example

We can argue that $R(3,3,3) = 17$. If we color each edge of K_{17} blue, red, or green, there will be a blue, red, or green triangle. The following is a 3-coloring of the edges of K_{16} that does not contain a monochromatic triangle.



Multicolor Ramsey's Theorem

Theorem (Multicolor Ramsey's Theorem)

There exists a least positive integer $R(s_1, \ldots, s_\ell)$ for which every ℓ-coloring of the edges of the complete graph on $R(s_1, \ldots, s_\ell)$ vertices contains some K_{s_i} of color i.

Proof of Schur's Theorem

Proof of Schur's Theorem.

Let c denote an ℓ-coloring of the natural numbers. Construct the complete graph K_n on the first $n = R(3, \ldots, 3)$ natural numbers. Define an ℓ-coloring c' of the edges of K_n by

$$c'(ij) = c(|j - i|).$$

There exists a monochromatic triangle made up of $u < v < w$ with

$$c(v - u) = c(w - v) = c(w - u)$$
$$(v - u) + (w - v) = (w - u)$$

□

Questions?

Thank you for your time!

If you have questions or comments about anything, do not hesitate to contact me at mgherman@caltech.edu.