Part 1

You might think this round is broken after solving some of these problems, but everything is intentional.

1. The number n can be represented uniquely as the sum of 6 distinct positive integers. Find n.

   Solution: The answers are 21, 22. If $n > 22$ then $n = 1+2+3+4+5+(n-15)$ and $n = 1+2+3+4+6+(n-16)$. Since $n-16 > 6$, these are both valid representations. If $n < 21$, then $n < 1+2+3+4+5+6$, so no representations are possible. It is straightforward to check that $21 = 1+2+3+4+5+6$ and $22 = 1+2+3+4+5+7$ are the only possible representations for these two numbers.

2. Let ABC be a triangle with $AB = BC$. The altitude from A intersects line BC at D. Suppose $BD = 5$ and $AC^2 = 1188$. Find AB.

   Solution: The answers are $22, 27$. Let x be the answer. There are two cases: either $\angle B$ is obtuse making D outside the triangle or it is acute making D on segment BC. In either case, note that $AD^2 = AB^2 - BD^2 = x^2 - 25$.

   Obtuse case: Then $CD = BD+BC = x+5$. So therefore $AC^2 = AD^2+CD^2 = x^2-25+(x+5)^2 = 2x^2+10x$. Setting this equal to 1188, we get $x^2+5x-594 = 0$. This has roots $22, -27$. So $x = 22$.

   Acute case: Then $CD = BC - BD = x - 5$. The same steps get us $AC^2 = 2x^2 - 10x$, so $x^2 - 5x - 594 = 0$. This has roots $27, -22$. So $x = 27$.

3. A lemonade stand analyzes its earning and operations. For the previous month it had a \$45 dollar budget to divide between production and advertising. If it spent k dollars on production, it could make $2k - 12$ glasses of lemonade. If it spent k dollars on advertising, it could sell each glass at an average price of $15 + 5k$ cents. The amount it made in sales for the previous month was \$40.50. Assuming the stand spent its entire budget on production and advertising, what was the absolute difference between the amount spent on production and the amount spent on advertising?

   Solution: The answers are $3, 21$. Assuming it spent x on production, the total amount made in sales was $(2x-12)(15+5(45-x))$, which we set equal to 4050 cents. This expands to $-10x^2+540x-2880 = 4050$, which can be simplified to $x^2 - 54x + 693 = 0$. This has roots $21, 33$. For $x = 21$, we have $45 - x = 24$ for a difference of 3. For $x = 33$, we have $45 - x = 12$ for a difference of 21.

4. Let A be the number of different ways to tile a $1 \times n$ rectangle with tiles of size 1×1, 1×3, and 1×6. Let B be the number of different ways to tile a $1 \times n$ rectangle with tiles of size 1×2 and 1×5, where there are 2 different colors available for the 1×2 tiles. Given that $A = B$, find n. (Two tilings that are rotations or reflections of each other are considered distinct.)

   Solution: The answers are $0, 8$. Letting A_n, B_n be the values for A, B for a particular n, we get $A_0 = B_0 = 1$, $A_n = A_{n-1} + A_{n-3} + A_{n-6}$ for $n \geq 1$, and $B_n = 2B_{n-2} + B_{n-5}$, where A_k, B_k are treated as 0 for any negative k. Obviously 0 is one value of n for which $A_n = B_n$. Computing out a few values shows $A_8 = B_8 = 16$. Past this A_n outgrows B_n; we omit the tedious details of showing no other solutions.

5. An integer $n \geq 0$ is such that n when represented in base 2 is written the same way as $2n$ is in base 5. Find n.

   Solution: The answers are $0, 3$. It is straightforward to check that each work. To show no others do, let the digit sequence be $d_k d_{k-1} \dots d_0$. Then we require that

   $$d_0(2-1) + d_1(4-5) + d_2(8-25) + \cdots + d_k(2^{k+1} - 3^k) = 0.$$

   Assuming not all the digits are 0, which gives $n = 0$, the only positive term on the LHS is $d_0(2-1)$, and the only other term with a coefficient of absolute value 1 is $d_1(4-5)$. So all the other digits are 0 and $d_0 = d_1 = 1$, which corresponds to $n = 3$.

6. Let x be a positive integer such that $3, \log_6(12x), \log_6(18x)$ form an arithmetic progression in some order. Find x.

   Solution: The answers are $8, 27$. For $\log_6(216), \log_6(12x), \log_6(18x)$ to be in arithmetic progression is the same as $216, 12x, 18x$ being in geometric progression. Note that $a < b < c$ are in geometric progression if and only if $ac = b^2$. If $18x$ is the median, $(18x)^2 = 216 \cdot 12x$, which gives $x = 8$. If 216 is the median, $216^2 = 12x \cdot 18x$, which gives $x = 6\sqrt{6}$ and can be thrown out since this does not make x an integer. If $12x$ is the median, $(12x)^2 = 216 \cdot 18x$, which gives $x = 27$. So $8, 27$ are the two possible integer values for x.

Part 2

Oops, it looks like there were some intentional printing errors and some of the numbers from these problems got removed. Any $\blacksquare$ that you see was originally some positive integer, but now its value is no longer readable. Still, if things behave like they did for Part 1, maybe you can piece the answers together.

1. The number n can be represented uniquely as the sum of $\blacksquare$ distinct positive integers. Find n.

2. Let ABC be a triangle with $AB = BC$. The altitude from A intersects line BC at D. Suppose $BD = \blacksquare$ and $AC^2 = 1536$. Find AB.

3. A lemonade stand analyzes its earning and operations. For the previous month it had a \$50 dollar budget to divide between production and advertising. If it spent k dollars on production, it could make $2k - 2$ glasses of lemonade. If it spent k dollars on advertising, it could sell each glass at an average price of $25 + 5k$ cents. The amount it made in sales for the previous month was \$$\blacksquare$. Assuming the stand spent its entire budget on production and advertising, what was the absolute difference between the amount spent on production and the amount spent on advertising?

4. Let A be the number of different ways to tile a $1 \times n$ rectangle with tiles of size $1 \times \blacksquare$, $1 \times \blacksquare$, and $1 \times \blacksquare$. Let B be the number of different ways to tile a $1 \times n$ rectangle with tiles of size $1 \times \blacksquare$ and $1 \times \blacksquare$, where there are $\blacksquare$ different colors available for the $1 \times \blacksquare$ tiles. Given that $A = B$, find n. (Two tilings that are rotations or reflections of each other are considered distinct.)

5. An integer $n \geq \blacksquare$ is such that n when represented in base 9 is written the same way as $2n$ is in base $\blacksquare$. Find n.

6. Let x be a positive integer such that $1, \log_{96}(6x), \log_{96}(\blacksquare x)$ form an arithmetic progression in some order. Find x.

Solution: The answers to the problems in order are:

$$3, 4, \quad 24, 32, \quad 0, 12, \quad 0, 3, \quad 12, 24, \quad 4, 32.$$

These problems have to all be solved together using the constraint seen from part 1: that each problem has two answers and an answer to a problem is also an answer to exactly one other problem.

Note that this solution will rigorously show the set of answers is unique and is far harder than the round needs to be. The part can be much more quickly solved with some educated guesses, although it is still not easy.

We begin by examining each problem in turn to see what we can learn at the outset.

1. The solution to part 1's problem 1 can be generalized to show that the answer to this problem is a triangular number and that triangular number plus one. Note that this triangular number is at least 1.

2. Let $BD = y$, and let x, z be the solutions to the acute and obtuse $\angle B$ cases respectively. First, note that $2x$ and $2z$ must be at least the length of AC by the triangle inequality, which implies $x, z > 19$ after some arithmetic. Using the work from part 1, we get in the $\angle B$ acute case that $1536 = x^2 - y^2 + (x-y)^2 = 2x(x-y)$, or $768 = x(x-y)$. In the obtuse case, we get $768 = z(z+y)$. Notice that if x is a solution to the first one, then $z = x - y$ is a solution to the second one, so the two solutions differ by y. Finally, and most importantly, $xz = 768$.

3. The total revenue is $f(x) = -10x^2 + 560x - 550$ where x is the amount spent on production. Since the linear term divided by the quadratic term is -56, we have $f(x) = f(56 - x)$.

4. This problem will have an answer of 0 no matter what the missing numbers are. This will be our starting point, but let's finish looking at the other problems.

5. Note that the first ■, being a positive integer, is at least 1, so n is at least 1. If n has one digit in base 9, then in another base that representation would also give a value of n. This implies $n = 2n$ or $n = 0$, which we know is false. So n has at least two digits in base 9, and both answers are at least 9.

6. We need for $96, 6x, \blacksquare x$ to be in geometric progression. Let $\blacksquare = a$. This means one of three cases depending on which is the middle number of the progression:

$$96^2 = 6ax_1^2, \quad 576x_2 = a^2x_2^2, \quad 96ax_3 = 36x_3^2,$$

$$x_1 = \frac{96}{\sqrt{6a}}, \quad x_2 = \left(\frac{24}{a}\right)^2, \quad x_3 = \frac{8a}{3}.$$

We need two of these possibilities to make x_i an integer, where a itself is an integer.

Now let us begin solving. As noted, we can start in problem 4 by finding 0 as an answer. We have verified that 0 cannot be an answer to any of problems 1, 2, 5, and it can't be an answer to 6 due to the domain of logarithms. So the other problem that 0 is an answer to is problem 3.

This means that one of the ways to achieve the ungiven sales revenue is if they had spent 25 dollars on each of production and advertising. Recall that $f(x) = f(56 - x)$, where $f(x)$ is the revenue from spending x dollars on production. So $f(25) = f(31)$ and a split of $31, 19$ for production and advertising will reap the same profit as $25, 25$. This means $31 - 19 = 12$ is the other answer to problem 3.

Now notice that the answers to problems 1, 3, 4, 5, 6 are all integers. Therefore, problem 2's answers must also be integers, or they can't match anything else. We know $xz = 768$ and $x > z > 19$. By looking at divisors of 768, we can find that the only pair of integers that satisfy both are $x = 32$, $z = 24$. So the two answers to problem 2 are $24, 32$.

At this point, we have four answers among problems 2 and 3, none of which are a triangular number or a triangular number plus one, which must be the two answers to problem 1. So the six distinct answers are $0, 12, 24, 32, \frac{n(n+1)}{2}, \frac{n(n+1)}{2} + 1$ where n is some positive integer. The first three problems each use these answers once, and the last three problems also do.

Notice that the only positive perfect squares we can have as answers are $1, 4$, which are $\frac{1\cdot 2}{2}$ and $\frac{2\cdot 3}{2} + 1$ respectively.

We now look at the possible values of a in problem 6. Consider the equations again:

$$x_1 = \frac{96}{\sqrt{6a}}, \quad x_2 = \left(\frac{24}{a}\right)^2, \quad x_3 = \frac{8a}{3}.$$

Only two need to result in integers. Suppose the first one does. Then $a = 6k^2$, which means $x_3 = (4k)^2$. This can't be 1 or 4, so it's impossible for both x_1, x_3 to be an integer, and x_2 is 1 or 4.

If $x_2 = 1$, we get $a = 24$, which makes $x_1 = 8$ and $x_3 = 64$, neither of which can be answers. So 1 is not an answer. Thus $x_2 = 4$ and $a = 12$. This makes x_1 not an integer, but $x_3 = 32$, which is the other answer.

Since 4 is an answer, the $\blacksquare$ in problem 1 is a 2 and its answers are 3, 4. Our six answers are $0, 3, 4, 12, 24, 32$. Problem 4's other answer and problem 5's two answers come from 3, 12, 24. We know problem 5's answers are both greater than 9, so they are $12, 24$. Then problem 4's second answer is 3, and there are an immense amount of ways to substitute numbers for the $\blacksquare$s to achieve this. This gets the answers listed at the top.