CHMMC 2021-2022

Rules

1. You have 120 minutes to complete the test.
2. You may collaborate with your other teammates on this round. Otherwise, **no other collaboration, computers, calculators, or other outside aid is permitted** besides what is listed below.
3. You are, however, permitted to use ruler and compass but **not** a protractor.
4. You can message an organizer **individually** in Slack to ask *only clarifying questions*. You may not receive a response if the question may help you solve the problems.
5. This round will count towards 60% of the group round score, the team round being worth the other 40%. This group round score in turn is half the team score.
6. The top 10 teams will be recognized, where the procedure the break ties has been detailed on the Chmmc website.
7. Your team needs to submit proofs to these problems. Directions for this are on the following page.

Directions

- Start your work on each problem at the top of a new page. "Parts" of multi-part problems may be started on the middle of a page.
- Each page of you submit must contain the problem number and team ID, written *clearly* at the top of the page. If you have multiple pages for a problem, number them and write the total number of pages for the problem (e.g. 1/3, 2/3, 3/3).
- Do not submit scratch work; you should only submit what you want the graders to see.
- Each problem is labelled a point value; more challenging problems may be labelled with more points. The total number of points available on this Round shall be labelled at the top of the Problems section. On multi-part problems, each part is also labelled a point value.
- For multi-part problems, you may solve the parts in any order and cite the assumed result of one part to solve another. Obviously, circular reasoning is not allowed.
- The word *compute* calls for an exact and simplified answer. Problems or parts of problems marked with compute shall be graded all-or-nothing. ALL OTHER PARTS OF THIS ROUND REQUIRE PROOF.
- Diagrams are not required for geometry problems.

Proof Round (33 Points Total)

Problem 1. *[4]* Find all ordered triples (a,b,c) of real numbers such that

$$(a-b)(b-c)+(b-c)(c-a)+(c-a)(a-b)=0.$$

Problem 2. *[4]* For any positive integer n, let $p(n)$ be the product of its digits in base-10 representation. Find the maximum possible value of $\frac{p(n)}{n}$ over all integers $n \geq 10$.

Problem 3. *[6]* Let $F(x_1,\ldots,x_n)$ be a polynomial with real coefficients in $n>1$ "indeterminate" variables $x_1,\ldots,x_n$. We say that F is *n-alternating* if for all integers $1 \leq i < j \leq n$,

$$F(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_n) = -F(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_n),$$

i.e. swapping the order of indeterminates x_i, x_j flips the sign of the polynomial. For example, $x_1^2x_2 - x_2^2x_1$ is 2-alternating, whereas $x_1x_2x_3 + 2x_2x_3$ is not 3-alternating.

Note: two polynomials $P(x_1,\ldots,x_n)$ and $Q(x_1,\ldots,x_n)$ are considered equal if and only if each monomial constituent $\alpha x_1^{k_1}\ldots x_n^{k_n}$ of P appears in Q with the same coefficient α, and vice versa. This is equivalent to saying that $P(x_1,\ldots,x_n)=0$ if and only if every possible monomial constituent of P has coefficient 0.

(1) *[2]* Compute a 3-alternating polynomial of degree 3.

(2) *[4]* Prove that the degree of any nonzero n-alternating polynomial is at least $\binom{n}{2}$.

Problem 4. *[5]* Show that for any three positive integers a,m,n such that m divides n, there exists an integer k such that $\gcd(a,m) = \gcd(a+km,n)$.

Problem 5. *[7]* Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(f(x)+f(y)^2) = f(x)^2 + y^2 f(y)^3.$$

Here $\mathbb{R}$ denotes the usual real numbers.

Problem 6. *[7]* Let ABC be an acute triangle with orthocenter H. A point $L \neq A$ lies on the plane of ABC such that $\overline{HL} \perp \overline{AL}$ and $LB:LC = AB:AC$. Suppose $M_1 \neq B$ lies on $\overline{BL}$ such that $\overline{HM_1} \perp \overline{BM_1}$ and $M_2 \neq C$ lies on $\overline{CL}$ such that $\overline{HM_2} \perp \overline{CM_2}$. Prove that $\overline{M_1M_2}$ bisects $\overline{AL}$.