Team Round Solutions

2018

1. Anita plays the following single-player game: She is given a circle in the plane. The center of this circle and some point on the circle are designated "known points". Now she makes a series of moves, each of which takes one of the following forms:

   (i) She draws a line (infinite in both directions) between two "known points"; or

   (ii) She draws a circle whose center is a "known point" and which intersects another "known point".

   Once she makes a move, all intersections between her new line/circle and existing lines/circles become "known points", unless the new/line circle is identical to an existing one. In other words, Anita is making a ruler-and-compass construction, starting from a circle.

   What is the smallest number of moves that Anita can use to construct a drawing containing an equilateral triangle inscribed in the original circle?

   Solution: $\boxed{5}$. Call the starting circle C, let P be the center of C, and let Q be the original known point on C. Anita can win in 5 moves as follows: draw the circle whose center is Q and which contains P, and let R, S denote the two intersections of the two circles; draw the line from P to Q, and let T denote the non-Q intersection of that line with C; and (in 3 moves) draw the lines $\overleftrightarrow{RS}$, $\overleftrightarrow{ST}$, and $\overleftrightarrow{TR}$. The resulting drawing contains the equilateral triangle $\triangle RST$ inscribed in C.

   To see that Anita cannot win in less than 5 moves, note that the first move must either be to draw the line $\overleftrightarrow{PQ}$ or to draw the circle with center Q containing P. After either of these moves, the resulting drawing does not contain 3 known points which form the vertices of an equilateral triangle inscribed in C. Thus she needs at least 1 more move to make all of the vertices. In addition, she needs at least additional 3 moves to draw the triangle, noting that no move which makes additional vertices can also draw one edge of the triangle, and no move which draws the triangle can create the additional vertices. Then in total, she needs at least $1 + 1 + 3 = 5$ moves.

2. Compute the sum $\sum_{n=1}^{200} \frac{1}{n(n+1)(n+2)}$.

   Solution: Decomposing via partial fractions, we have $\frac{1}{n(n+1)(n+2)} = \frac{1}{2}(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2})$, so the sum telescopes, giving $\sum_{n=1}^{200} \frac{1}{n(n+1)(n+2)} = \frac{1}{2}(1 - \frac{1}{2} - \frac{1}{201} + \frac{1}{202}) = \boxed{\frac{5075}{20301}}$.

3. Let p be the third-smallest prime number greater than 5 such that:

   - $2p + 1$ is prime, and
   - $5^p \not\equiv 1 \pmod{2p+1}$.

Find p.

Solution: Note that whenever $2p+1$ is prime, $5^{2p} \equiv 1 \pmod{2p+1}$ by Fermat's little theorem. Hence we just need to figure out when $5^p \not\equiv 1 \pmod{2p+1}$. By Euler's criterion, $5^p \equiv \left(\frac{5}{2p+1}\right)$, and by quadratic reciprocity, $\left(\frac{5}{2p+1}\right) = \left(\frac{2p+1}{5}\right)$. Since p and $2p+1$ are prime and not 5, we see that the only possibilities are $p \equiv 1, 3, 4 \pmod 5$, leading to $2p+1 \equiv 2, 3, 4 \pmod 5$, respectively. Out of $2, 3, 4$, only 2 and 3 are not squares modulo 5, so $5^p \not\equiv 1 \pmod{2p+1}$ iff $p \equiv 1$ or $3 \pmod 5$.

Now we enumerate all $p > 5$ such that $2p+1$ is prime and $p \equiv 1, 3 \pmod 5$: 11, 23, $\boxed{41}$.

4. If Percy rolls a fair six-sided die until he rolls a 5, what is his expected number of rolls, given that all of his rolls are prime?

   Solution: For a given $k \geq 1$, the probability p_k that there are exactly k rolls and all rolls are prime is $\frac{1}{6}(\frac{1}{3})^{k-1}$ (this happens if rolls $1, \ldots, k-1$ are in $\{2, 3\}$, and the last is 5). Then the probability p that all rolls are prime is the sum $p = \sum_{k=1}^{\infty} p_k = \frac{1}{6}\sum_{k=1}^{\infty}(\frac{1}{3})^{k-1} = \frac{1}{4}$, so the expected number of rolls, given that all rolls are prime is $\frac{1}{p}\sum_{k=1}^{\infty} kp_k = \frac{2}{3}\sum_{k=1}^{\infty} k(\frac{1}{3})^{k-1} = \boxed{3/2}$.

5. Let $\triangle ABC$ be a right triangle such that $AB = 3$, $BC = 4$, $AC = 5$. Let point D be on AC such that the incircles of $\triangle ABD$ and $\triangle BCD$ are mutually tangent. Find the length of BD.

   Solution: We can divide each of the edges BA, BC, AD, CD, BD in two at the points of tangency of the two incircles; since the two incircles are tangent at the same point on BD, the two divisions of BD agree, so we can write $BA = b+a$, $BC = b+c$, $AD = a+d$, $CD = c+d$, and $BD = b+d$, for some a, b, c, d. Drop the altitude from B to AC, and let the point at the base be E. Writing $x = ED$, we have $AD = AE + ED = \frac{9}{5} + x$, $CD = CE - ED = \frac{16}{5} - x$, and $BD = \sqrt{BE^2 + ED^2} = \sqrt{(\frac{12}{5})^2 + x^2}$. To find x, we can express $2b$ in two ways as $2b = BC + BD - CD = BA + BD - AD$. This gives $BC - CD = BA - AD$, hence $4 - (\frac{16}{5} - x) = 3 - (\frac{9}{5} + x)$, and it follows that $x = \frac{1}{5}$, giving $BD = \sqrt{(\frac{12}{5})^2 + (\frac{1}{5})^2} = \boxed{\sqrt{145}/5}$.

6. Karina has a polynomial $p_1(x) = x^2 + x + k$, where k is an integer. Noticing that p_1 has integer roots, she forms a new polynomial $p_2(x) = x^2 + a_1x + b_1$, where a_1 and b_1 are the roots of p_1 and $a_1 \geq b_1$. The polynomial p_2 also has integer roots, so she forms a new polynomial $p_3(x) = x^2 + a_2x + b_2$, where a_2 and b_2 are the roots of p_2 and $a_2 \geq b_2$. She continues this process until she reaches $p_7(x)$ and finds that it does not have integer roots. What is the largest possible value of k?

   Solution: p_1 will have integer roots only if $-k = n(n-1)$, in which case $p_2(x) = x^2 + (n-1)x - n$ and $p_3(x) = x^2 + x - n$. Similarly, $p_5(x) = x^2 + x - n_2$ where $n = n_2(n_2 - 1)$ and $p_7(x) = x^2 + x - n_3$ where $n_2 = n_3(n_3 - 1)$. Now the smallest possible value for n_3 is 3 (if it were 0 or 1 then we wouldn't actually reach p_7 and if it were 2 we could go on forever), so $n_2 = 6$, $n = 30$, and $-k = 870$, hence the largest value of k is $\boxed{-870}$.

7. For a positive number n, let $g(n)$ be the product of all $1 \leq k \leq n$ such that $\gcd(k, n) = 1$, and say that $n > 1$ is *reckless* if n is odd and $g(n) \equiv -1 \pmod n$. Find the number of reckless numbers less than 50.

Solution: We will prove that the reckless numbers are exactly the odd prime powers, from which it follows that the answer is $\boxed{18}$, since there are 18 odd prime powers less than 50.

First note that every k in the product $g(n)$ has a unique inverse k' mod n, which is also in the product. We can group all elements of the product into pairs of inverses, except those which are their own inverses, i.e. those which have $k^2 \equiv 1 \pmod{n}$. The product of each such pair cancels to 1 mod n, so it follows that $g(n) \equiv \prod_{k^2\equiv 1} k \pmod{n}$.

Now, with induction on m, we can prove for an odd prime power p^m that the only solutions to $k^2 \equiv 1 \pmod{p^m}$ are $k \equiv \pm 1 \pmod{p^m}$. First, in the case $m = 1$, we have $k^2 - 1 = (k-1)(k+1)$, so since p is prime the only solutions to $(k-1)(k+1) \equiv 0 \pmod{p}$ are $k \equiv \pm 1 \pmod{p}$. Assuming the statement holds for some $m \geq 1$, consider the equation $k^2 \equiv 1 \pmod{p^{m+1}}$. Any solution k must have $k \equiv \pm 1 \pmod{p^m}$ by the inductive hypothesis, so $k \equiv ap^m \pm 1 \pmod{p^{m+1}}$ for some a, giving $1 \equiv k^2 \equiv 1 \pm 2ap^m \pmod{p^{m+1}}$. This can only hold if $a \equiv 0 \pmod{p}$, meaning $k \equiv \pm 1 \pmod{p^{m+1}}$, as desired.

From this it follows that $g(p^m) \equiv (1)(-1) \equiv -1 \pmod{p^m}$ for any odd prime power p^m, so all such p^m are reckless. Now, consider an odd composite n, so $n = p_1^{m_1} \cdots p_r^{m_r}$ with $r > 1$. Any k with $k^2 \equiv 1 \pmod{n}$ has $k^2 \equiv 1 \pmod{p_i^{m_i}}$ for each i, hence $k \equiv k_i \pmod{p_i^{m_i}}$ for some $k_i \in \{-1, 1\}$. Conversely, for any choice of $k_1, \ldots, k_r \in \{-1, 1\}$, by the Chinese Remainder Theorem there is a unique $1 \leq k \leq n$ with $k \equiv k_i \pmod{p_i^{m_i}}$ for each i, and this k necessarily has $k^2 \equiv 1 \pmod{p_i^{m_i}}$ for each i, so again by the CRT we have $k^2 \equiv 1 \pmod{n}$. The values of k with $k^2 \equiv 1 \pmod{n}$ are thus in direct correspondence with choices of $k_1, \ldots, k_r \in \{-1, 1\}$, so in particular, there are 2^r such k, and for each i, 2^{r-1} of these k have $k \equiv 1 \pmod{p_i^{m_i}}$ and 2^{r-1} have $k \equiv -1 \pmod{p_i^{m_i}}$. Thus $g(n) \equiv (1)^{2^{r-1}}(-1)^{2^{r-1}} \equiv 1 \pmod{p_i^{m_i}}$ for each i, so $g(n) \equiv 1 \pmod{n}$, meaning n is not reckless.

8. Find the largest positive integer n that cannot be written as $n = 20a + 28b + 35c$ for nonnegative integers a, b, and c.

   Solution: Generalizing the problem: for given relatively prime p, q, r, we want to find the largest positive integer n that cannot be written as $n = apq + bpr + cqr$ for $a, b, c \geq 0$. Let S be the set of numbers n which can be written as $n = apq + bpr$ for $a, b \geq 0$. Note all such numbers are multiples of p. By the "Chicken McNugget Theorem", the largest number which cannot be written in the form $aq + br$ for $a, b \geq 0$ is $qr - q - r$, so the largest multiple of p which cannot be written in the form $apq + bpr$ is $p(qr - q - r)$, and all larger multiples of p can be written in this form. Thus $p(qr - q - r) \notin S$, while $pk \in S$ for all $k > p(qr - q - r)$.

   The rest of the proof is similar to that of the Chicken McNugget Theorem. Let T be the set of numbers n which can be expressed as $n = apq + bpr + cqr$ for $a, b, c \geq 0$. Any element of T can necessarily be written this way with $c < p$, since otherwise we can write $c = pd + c'$ for $d \geq 0$ and $0 \leq c' < p$, hence $n = apq + bpr + (pd + c')qr = (a + rd)pq + bpr + c'qr$. Thus T is the set of numbers which can be written as $n = s + cqr$ for $s \in S$ and $0 \leq c < p$. Any such expression is in fact unique; this can be seen by noting that $c \equiv q^{-1}r^{-1}n \pmod{p}$.

   Thus for a given $0 \leq \ell < p$, for $n \equiv \ell \pmod{p}$ we have $n \in T \iff n - cqr \in S$, where c is such that $0 \leq c < p$ and $c \equiv q^{-1}r^{-1}\ell \pmod{p}$. This means that the largest n with $n \equiv \ell$ and $n \notin T$ is $p(qr - q - r) + cqr$, for the same value of c as above. Since

c ranges over $0, 1, \dots, p-1$ as ℓ ranges over $0, 1, \dots, p-1$, it follows that the largest value of n not in T is $n = p(qr - q - r) + (p-1)qr = 2pqr - pq - pr - qr$. In our case, $p = 4$, $q = 5$, $r = 7$, so this value is $\boxed{197}$.

9. Say that a function $f : \{1, 2, \dots, 1001\} \to \mathbb{Z}$ is *almost polynomial* if there is a polynomial $p(x) = a_{200}x^{200} + \cdots + a_1 x + a_0$ such that each a_n is an integer with $|a_n| \leq 201$, and such that $|f(x) - p(x)| \leq 1$ for all $x \in \{1, 2, \dots, 1001\}$. Let N be the number of almost polynomial functions. Compute the remainder upon dividing N by 199.

   Solution: Let P be the set of all polynomials of the desired form, that is, all polynomials $p(x) = a_{200}x^{200} + \cdots + a_1 x + a_0$ such that each a_n is an integer with $|a_n| \leq 201$. Also, let A be the set of pairs (f, p) of a function f and a polynomial p which satisfy the given condition (meaning in particular $p \in P$), so a function f is almost polynomial iff there is a polynomial p such that $(f, p) \in A$. Suppose that f is almost polynomial and there are two polynomials p_1, p_2 which satisfy the condition for f, i.e. $(f, p_1), (f, p_2) \in A$. Then for all $x \in \{1, 2, \dots, 1001\}$, since $|f(x) - p_1(x)| \leq 1$ and $|f(x) - p_2(x)| \leq 1$, we must have $|p_1(x) - p_2(x)| \leq 2$, so $p_1(x) - p_2(x) \in \{-2, -1, 0, 1, 2\}$. By the pigeonhole principle, there is some $c \in \{-2, -1, 0, 1, 2\}$ such that $p_1(x) - p_2(x) = c$ for at least 201 distinct values of x, but since $p_1(x) - p_2(x) - c$ is a polynomial of degree at most 200, this is impossible unless $p_1(x) - p_2(x) - c = 0$. Thus any two polynomials which satisfy the condition for the same function differ by a constant (between -2 and 2).

   For a given almost polynomial f, let p_0 be the polynomial, among all those with $(f, p) \in A$, with the smallest constant term. Then by the above, the only other possible choices of p with $(f, p) \in A$ are $p(x) = p_0(x) + 1$ and $p(x) = p_0(x) + 2$. Note that if $(f, p_0 + 2) \in A$, then since $|f(x) - p_0(x)| \leq 1$ and $|f(x) - (p_0(x) + 2)| \leq 1$, we must have $f(x) = p_0(x) + 1$ for all x, so in particular, $(f, p_0 + 1) \in A$ as well. Thus the three possible cases for the set of polynomials p with $(f, p) \in A$ are (i) p_0, (ii) $p_0, p_0 + 1$, and (iii) $p_0, p_0 + 1, p_0 + 2$. Now, consider the sum

   $$S = \sum_{p \in P} |\{f : (f, p) \in A\}| - \sum_{p \in P} |\{f : (f, p), (f, p+1) \in A\}|.$$

   For a function f in case (i), f is counted once in the first sum, and not at all in the second sum. If f falls into case (ii), f is counted twice in the first sum (at $p = p_0, p_0 + 1$) and once in the second sum (at $p = p_0$). If f falls into case (iii), it is counted three times in the first sum (at $p = p_0, p_0 + 1, p_0 + 2$) and twice in the second sum (at $p = p_0, p_0 + 1$). Thus each almost polynomial function f is counted exactly once in S, so $S = N$.

   Now to find N, we have to evaluate these two sums. The first is straightforward: for a given $p \in P$, the number of f with $(f, p) \in A$ is 3^{1001}, since a choice of f simply corresponds to a choice of $f(x) \in \{p(x) - 1, p(x), p(x) + 1\}$ for each $x \in \{1, 2, \dots, 1001\}$, so the first sum comes out to $3^{1001}|P| = 3^{1001} \cdot 403^{201}$. The second sum is slightly more complicated. For $p \in P$, if $p + 1 \notin P$, then there can be no f with $(f, p), (f, p+1) \in A$, so the summand corresponding to p is zero. Now if $p \in P$ is such that $p + 1 \in P$ as well, then there are exactly 2^{1001} choices of f with $(f, p), (f, p + 1) \in A$, since we must have $f(x) \in \{p(x), p(x) + 1\}$ for all x. But the $p \in P$ such that $p + 1 \in P$ as well are exactly those with constant term not equal to 201, so the number of such p is $403^{200} \cdot 402$, and thus the second sum comes out to $2^{1001} \cdot 403^{200} \cdot 402$.

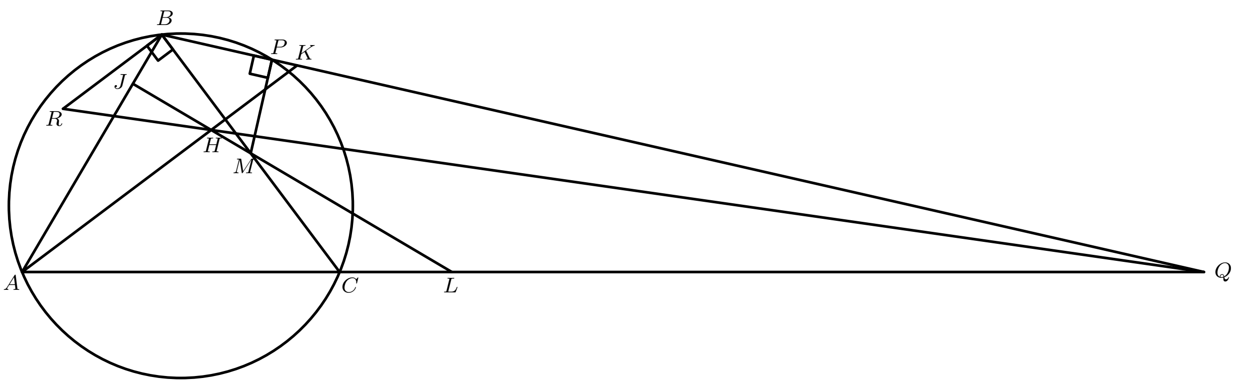
   Thus, evaluating N mod 199, using the fact that 199 is prime, together with Fermat's little theorem, we have

$$\begin{aligned} N &= 3^{1001} \cdot 403^{201} - 2^{1001} \cdot 403^{200} \cdot 402 \\ &\equiv 3^{1001} \cdot 5^{201} - 2^{1001} \cdot 5^{200} \cdot 4 \\ &\equiv 3^{11} \cdot 5^{3} - 2^{11} \cdot 5^{2} \cdot 4 \\ &\equiv 3 \cdot 243^{2} \cdot 5^{3} - 2^{10} \\ &\equiv 3 \cdot 44^{2} \cdot 5^{3} - 29 \\ &\equiv 3 \cdot 11^{2} \cdot 10 - 29 \\ &\equiv \boxed{19} \pmod{199} \end{aligned}$$

10. Let ABC be a triangle such that $AB = 13, BC = 14, AC = 15$. Let M be the midpoint of BC and define $P \neq B$ to be a point on the circumcircle of ABC such that $BP \perp PM$. Furthermore, let H be the orthocenter of ABM and define Q to be the intersection of BP and AC. If R is a point on HQ such that $RB \perp BC$, find the length of RB.

    Solution: Interestingly, the answer does not depend on the point M. In fact, we will show that R satisfies $CR \perp AB$. Knowing this greatly simplifies things as you no longer need to know how R is constructed through P, H, and Q. In essence, R is an easily constructible point embellished by lots of "fluff."

    Now to prove $CR \perp AB$, let J be the intersection of HM with AB, L the intersection of HM and AC, and K the intersection of AH and BP.



    Because H is the orthocenter of ABM, $MJ \perp AB$, which implies B, J, M, P are concyclic. This means

$$\angle BAH = 90 - \angle B = \angle JMB = \angle JPB,$$

    and so

$$\angle JAK = \angle BAH = \angle JPB = 180 - \angle JPK,$$

    which implies A, J, P, K are concyclic. Observe that

$$\angle AKJ = \angle APJ = \angle APB - \angle JPB = \angle C - (90 - \angle B) = 90 - \angle A = \angle ALJ.$$

    This proves that A, J, K, L are concyclic.

Thus, $\angle JLK = \angle JAK = 90 - \angle B = \angle JMB$, which proves $BM \parallel KL$, that is, $BC \parallel KL$. Now because $RB \parallel HK$, we conclude that triangles RBC and HKL are homothetic with respect to Q, and thus $CR \parallel JL$, which means $CR \perp AB$.

The rest is straightforward. Let E be the intersection of CR and AB, so in particular, E is the base of the altitude from C to AB. The area of ABC is 84, which gives $EC = \frac{168}{13}$, hence $EB = \frac{70}{13}$. Using the fact that $\frac{EB}{EC} = \frac{RB}{BC}$, we have $RB = \boxed{35/6}$.