Individuals Tiebreaker 2022-2023 Solutions

Problem 1. Yor and Fiona are playing a match of tennis against each other. The first player to win 6 games wins the match (while the other player loses the match). Yor has currently won 2 games, while Fiona has currently won 0 games. Each game is won by one of the two players: Yor has a probability of $\frac{2}{3}$ to win each game, while Fiona has a probability of $\frac{1}{3}$ to win each game. Then, $\frac{m}{n}$ is the probability Fiona wins the tennis match, for relatively prime integers m,n. Compute m.

Proposed by Brian Yang

Solution: $\boxed{835}$.

Remark it will take at most 9 games to decide the winner of the tennis match. To this end, imagine Yor and Fiona play 9 more consecutive games of tennis, continuing to play even if the winner is decided. If Yor wins at least 4 games out of 9, then Yor will have won at least 6 games in the (original) tennis match, hence winning the tennis match. Likewise, if Yor wins no more than 3 games out of 9, then Fiona will have won at least 6 games in the tennis match, hence Fiona wins the tennis match. Hence, the desired probability is

$$\frac{1}{3^9}\binom{9}{0}+\frac{2}{3^9}\binom{9}{1}+\frac{2^2}{3^9}\binom{9}{2}+\frac{2^3}{3^9}\binom{9}{3}=\frac{835}{3^9}.$$

The answer is $\boxed{835}$.

Problem 2. Jonathan and Eric are standing one kilometer apart on a large, flat, empty field. Jonathan rotates an angle of $\theta = 120^\circ$ counterclockwise around Eric, then Eric moves half of the distance to Jonathan. They keep repeating the previous two movements in this order. After a very long time, their locations approach a point P on the field. What is the distance, in kilometers, from Jonathan's starting location to P?

Proposed by Mathus Leungpathomaram

Solution: $\boxed{\frac{2\sqrt{21}}{7}}$.

Let J_n and E_n be Jonathan's and Eric's positions in the complex plane after n sets of movements, with $J_0 = 0$ and $E_0 = 1$ being their starting positions. Observe that $E_n - J_{n+1} = e^{\frac{2\pi i}{3}} \cdot (E_n - J_n)$, and $E_{n+1} - J_{n+1} = \frac{1}{2} \cdot (E_n - J_{n+1})$. Combining these, we have $E_{n+1} - J_{n+1} = \frac{e^{\frac{2\pi i}{3}}}{2} \cdot (E_n - J_n)$. Furthermore, J_n, E_n, J_{n+1} forms an isosceles triangle with a 120° angle in the complex plane, so that $J_{n+1} - J_n = \sqrt{3}e^{-\frac{\pi i}{6}} \cdot (E_n - J_n)$, so $J_{n+2} - J_{n+1} = \frac{e^{\frac{2\pi i}{3}}}{2} \cdot (J_{n+1} - J_n)$. By induction, this means $J_{n+1} - J_n = \left(\frac{e^{\frac{2\pi i}{3}}}{2}\right)^n \cdot (J_1 - J_0)$ for all $n \geq 0$. Computing $|P|$:

$$\begin{aligned} P &= J_0 + \sum_{n=0}^{\infty}(J_{n+1} - J_n) = (J_1 - J_0) \cdot \sum_{n=0}^{\infty}\left(\frac{e^{\frac{2\pi i}{3}}}{2}\right)^n \\ &= \sqrt{3}e^{-\frac{\pi i}{6}} \cdot \frac{1}{1 - \frac{e^{\frac{2\pi i}{3}}}{2}} = \frac{4\sqrt{3}e^{-\frac{\pi i}{6}}}{4 - (-1 + \sqrt{3}i)} = \frac{4\sqrt{3}e^{-\frac{\pi i}{6}}}{5 - \sqrt{3}i}. \\ |P| &= \frac{4\sqrt{3}}{\sqrt{5^2 + \left(\sqrt{3}\right)^2}} = \frac{4\sqrt{3}}{\sqrt{28}} = \boxed{\frac{2\sqrt{21}}{7}}. \end{aligned}$$

Remark: This problem can also be solved by using the facts $J_1P = \frac{J_0P}{2}, \angle J_0PJ_1 = 120^\circ$, and then using Law of Cosines on $\triangle J_0J_1P$.

Problem 3. Suppose that a,b,c are complex numbers with $a+b+c=0, |abc|=1, |b|=|c|$, and

$$\frac{9-\sqrt{33}}{48} \le \cos^2\left(\arg\left(\frac{b}{a}\right)\right) \le \frac{9+\sqrt{33}}{48}.$$

Find the maximum possible value of $|-a^6+b^6+c^6|$.

Proposed by August Chen

Solution: $\boxed{9-4\sqrt{3}}$.

Say $|a| = r^2$ for some $r > 0$, so $|b| = |c| = \frac{1}{r}$. By a suitable rotation we can assume $a = r^2$ since all we care about is the final magnitude, so write $b = \frac{1}{r}e^{\theta_1 i}, c = \frac{1}{r}e^{\theta_2 i}$. We need $0 = a+b+c = r^2 + \frac{1}{r}(e^{\theta_1 i} + e^{\theta_2 i})$, and this forces $\theta_2 = -\theta_1$. Let $\theta = \theta_1$; notice that this equals $\arg(\frac{b}{a})$. Now we have $\frac{2}{r}\cos\theta + r^2 = 0$, i.e. $r^3 = -2\cos\theta$. Now, $-a^6+b^6+c^6 = -r^{12} + \frac{2}{r^6}\cos 6\theta = -16\cos^4\theta + \frac{\cos 6\theta}{2\cos^2\theta}$. We have $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, so $\cos 6\theta = 2(16\cos^6\theta - 24\cos^4\theta + 9\cos^2\theta) - 1$. This means

$$\begin{aligned} |-a^6+b^6+c^6| &= \left|\frac{-32\cos^6\theta + 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1}{2\cos^2\theta}\right| \\ &= \left|24\cos^2\theta + \frac{1}{2\cos^2\theta} - 9\right| \end{aligned}$$

For $x > 0$, notice $24x + \frac{1}{2x} \le 9$ if and only if $48x^2 - 18x + 1 \le 0$. The solution to this quadratic inequality is $\frac{9-\sqrt{33}}{48} \le x \le \frac{9+\sqrt{33}}{48}$. But this is exactly the condition on $\cos^2\theta$! Thus, $24\cos^2\theta + \frac{1}{2\cos^2\theta} \le 9$, so

$$\left|24\cos^2\theta + \frac{1}{2\cos^2\theta} - 9\right| = 9 - \left(24\cos^2\theta + \frac{1}{2\cos^2\theta}\right) \le 9 - 2\sqrt{12} = \boxed{9-4\sqrt{3}}$$

where the last step follows by AM-GM; this is justified as the equality case is $\cos^2\theta = \sqrt{\frac{1}{48}} \in (\frac{9-\sqrt{33}}{48}, \frac{9+\sqrt{33}}{48})$.

Problem 4. Let ABC be a triangle with $AB = 4, BC = 5, CA = 6$. Triangles APB and CQA are erected outside ABC such that $AP = PB$, $\overline{AP} \perp \overline{PB}$ and $CQ = QA$, $\overline{CQ} \perp \overline{QA}$. Pick a point X uniformly at random from segment $\overline{BC}$. What is the expected value of the area of triangle PXQ?

Proposed by Brian Yang

Solution: $\boxed{\dfrac{52+15\sqrt{7}}{8}}$.

Observe that $[PXQ]$ is directly proportional to the length l of the altitude of X onto $\overline{PQ}$, and that l is a linear function of BX. Hence, the distribution of $[PXQ]$ is uniform, and so it attains its expectation value precisely when X is the midpoint of $\overline{BC}$. Henceforth assume X is the midpoint of $\overline{BC}$; we need to compute $[PXQ]$.

Let D and E be the reflections of B and C over P and Q, respectively. Since $\triangle APB$ and $\triangle CQA$ are isosceles triangles with right angles at $\angle BPA$ and $\angle AQC$, respectively, $\triangle DAB$ and $\triangle EAC$ are also isosceles right triangles (both of which are outside of $\triangle ABC$), each of which has a right angle at vertex A. In particular, there is a 90°

rotation around point A which maps C to E and D to B, implying that $\overline{CD} \perp \overline{EB}$ and $CD = BE$. Then by the midline theorem, $\triangle PXQ$ is a right triangle with $PX = \frac{CD}{2} = \frac{BE}{2} = XQ$.
Let $\theta = \angle CAD = \angle CAB + \angle BAD = \angle CAB + 90^\circ$. Then, $\cos\theta = \cos(\angle CAB + 90^\circ) = -\sin(\angle CAB)$. Applying the Law of Cosines in $\triangle ABC$, we get $\cos(\angle CAB) = \frac{9}{16}$, which means $\cos\theta = -\sin(\angle CAB) = -\frac{5\sqrt{7}}{16}$. Now by the Law of Cosines in $\triangle CAD$:

$$CD^2 = CA^2 + AD^2 - 2 \cdot CA \cdot AD \cdot \cos\theta = 4^2 + 6^2 + 2 \cdot 4 \cdot 6 \cdot \frac{5\sqrt{7}}{16} = 52 + 15\sqrt{7},$$

using the fact $AD = AB = 4$. The answer is $[PXQ] = \frac{1}{2}(\frac{CD}{2})^2 = \frac{CD^2}{8} = \boxed{\frac{52+15\sqrt{7}}{8}}$.