CHMMC 2020-2021

Individual Round Solutions

1. A right triangle ABC is inscribed in the circular base of a cone. If two of the side lengths of ABC are 3 and 4, and the distance from the vertex of the cone to any point on the circumference of the base is 3, then the minimum possible volume of the cone can be written as $\frac{m\pi\sqrt{n}}{p}$, where m, n, and p are positive integers, m and p are relatively prime, and n is squarefree. Find $m+n+p$.

Solution: 12

By the Pythagorean Theorem, the side lengths of ABC are either $3, \sqrt{7}, 4$ or $3, 4, 5$. The slant height of the cone is 3, and the radius of the circular base is either 2 or $\frac{5}{2}$, so the height of the cone, by the Pythagorean Theorem, is either $\sqrt{5}$ or $\frac{\sqrt{11}}{2}$, respectively. The volume of the cone, given by the formula $\frac{h}{3} \cdot \pi r^2$, is either $\frac{\pi}{3} \cdot 2^2 \cdot \sqrt{5}$ or $\frac{\pi}{3} \cdot \frac{5^2}{2^2} \cdot \frac{\sqrt{11}}{2}$. We see that the former number is smaller than the latter, so the minimum volume of the cone is

$$\frac{1}{3} \cdot \sqrt{5} \cdot 2^2 \cdot \pi = \frac{4\pi\sqrt{5}}{3}.$$

The requested sum is $\boxed{12}$.

2. Caltech's 900 students are evenly spaced along the circumference of a circle. How many equilateral triangles can be formed with at least two Caltech students as vertices?

Solution: 808500

There are $\binom{900}{2} = 404550$ pairs of students, and each pair can have an equilateral triangle on either side of the line connecting them. However, some of the triangles have been triple-counted (i.e. when the third vertex is another student). This happens $\frac{900}{3} = 300$ times for each inscribed equilateral triangle. This yields $404550 \cdot 2 - 2 \cdot 300 = \boxed{808500}$ equilateral triangles.

3. A *Beaver-number* is a positive 5 digit integer whose digit sum is divisible by 17. Call a pair of *Beaver-numbers* differing by exactly 1 a *Beaver-pair*. The smaller number in a *Beaver-pair* is called an *MIT Beaver*, while the larger number is called a *CIT Beaver*. Find the positive difference between the largest and smallest *CIT Beavers* (over all *Beaver-pairs*).

Solution: 79200

If k is an *MIT Beaver*, then $k+1$, the *CIT Beaver*, must have two carryovers in addition. This is because the digit sum of k minus the digit sum of $k+1$ must be a multiple of 17. Thus, $k = \overline{abc99}$ where $a \neq 0, c \neq 9$. Clearly, $a+b+c = 16$. By trying to maximize and minimize with the leftmost digits, we get that the largest and smallest *MIT Beavers* are 97099 and 17899, respectively. The difference of these two *MIT Beavers* is the same as the difference of the largest and smallest *CIT Beavers*, which is just $\boxed{79200}$.

4. Let $P(x) = x^3 - 6x^2 - 5x + 4$. Suppose that y and z are real numbers such that

$$zP(y) = P(y-n) + P(y+n)$$

for all reals n. Evaluate $P(y)$.

Solution: -22

We claim that $z = 2$. By taking $n = 0$, we have that

$$zP(y) = 2P(y),$$

so $z = 2$ or $P(y) = 0$. Assume on the contrary that $P(y) = 0$. Observe that $P(-10) < 0, P(0) = 4, P(2) = -22, P(10) > 0$. Hence, by the Intermediate Value Theorem on $[-10, 0], [0, 2]$, and $[2, 10]$, the three complex roots of $P(x)$ are all real. Thus, if we take n such that $P(y - n) = 0$, then we must also have $P(y + n) = 0$. So the three roots of $P(x)$ are $y - n, y, y + n$. By Vieta's formulas, $3y = 6$ and $y = 2 \implies P(2) = -22$, which is a contradiction. Thus, $z = 2$.

Since $z = 2$, observe that we can take $(y, P(y))$ as the midpoint of the segment between $(y - n, P(y - n))$ and $(y + n, P(y + n))$ for $z = 2$. Thus, there is $Q(x) = mx + b$ such that the roots of $P(x) - Q(x)$ are $y - n$, y and $y + n$. The x^2 coefficient of $P(x) - Q(x)$ is -6, so by Vieta's formulas, $3y = 6$ and $y = 2 \implies P(2) = \boxed{-22}$.

5. Let S be the sum of all positive integers n such that $\frac{3}{5}$ of the positive divisors of n are multiples of 6 and n has no prime divisors greater than 3. Compute $\frac{S}{36}$.

Solution: 2345

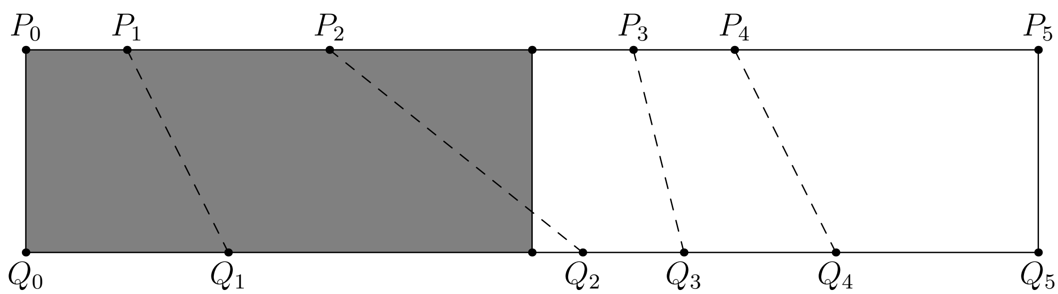
For a positive integer $n = 2^a 3^b$, the fraction of positive divisors divisible by 2 is given by $\frac{a}{a+1}$, as we have $a + 1$ total choices for the exponent of 2 in a positive divisor of n, a of which yields a multiple of 2. Similarly, the fraction of positive divisors divisible by 3 is given by $\frac{b}{b+1}$. Hence, the fraction of positive divisors that are a multiple of 6 is $\frac{ab}{(a+1)(b+1)} = \frac{3}{5}$. This equation simplifies to $2ab - 3a - 3b - 3 = 0$. Now, by Simon's Favorite Factoring Trick, we have that

$$4ab - 6a - 6b - 6 = 0 \implies 4ab - 6a - 6b + 9 = 15 \implies (2a - 3)(2b - 3) = 15,$$

yielding nonnegative integer solutions $(a, b) = (9, 2), (4, 3), (3, 4), (2, 9)$. Thus,

$$\frac{S}{36} = \frac{2^9 3^2 + 2^4 3^3 + 2^3 3^4 + 2^2 3^9}{36} = \boxed{2345}.$$

6. Let $P_0P_5Q_5Q_0$ be a rectangular chocolate bar, one half dark chocolate and one half white chocolate, as shown in the diagram below. We randomly select 4 points on the segment P_0P_5, and immediately after selecting those points, we label those 4 selected points P_1, P_2, P_3, P_4 from left to right. Similarly, we randomly select 4 points on the segment Q_0Q_5, and immediately after selecting those points, we label those 4 points Q_1, Q_2, Q_3, Q_4 from left to right. The segments $P_1Q_1, P_2Q_2, P_3Q_3, P_4Q_4$ divide the rectangular chocolate bar into 5 smaller trapezoidal pieces of chocolate. The probability that exactly 3 pieces of chocolate contain both dark and white chocolate can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Solution: 39

Let p be the number of points P_i $(i \geq 1)$ on the dark chocolate side of the bar, and q be the number of points Q_i $(i \geq 1)$ on the dark chocolate side of the bar. Then, if n is the number of pieces containing both dark and white chocolate, we can see with inspection that

$$n = |p - q| + 1.$$

Since we want $n = 3$, the ordered pairs (p, q) that work are $(4, 2), (3, 1), (2, 0), (2, 4), (1, 3)$, and $(0, 2)$. Since each point is randomly chosen, the probability of each $p \in \{0, 1, 2, 3, 4\}$ is $\frac{\binom{4}{p}}{16}$. Hence, the probability that exactly 3 pieces of chocolate contain both dark and white chocolate is

$$2\left(\frac{\binom{4}{4}\binom{4}{2}}{16^2} + \frac{\binom{4}{3}\binom{4}{1}}{16^2} + \frac{\binom{4}{2}\binom{4}{0}}{16^2}\right) = \frac{7}{32},$$

where we multiply by 2 due to the symmetry of binomial coefficients. The answer is $\boxed{39}$.

7. Given 10 points on a plane such that no three are collinear, we connect each pair of points with a segment and color each segment either red or blue. Assume that there exists some point A among the 10 points such that:

(1) There is an odd number of red segments connected to A

(2) The number of red segments connected to each of the other points are all different

Find the number of red triangles (i.e, a triangle whose three sides are all red segments) on the plane.

Solution: 30

The problem can be quickly solved by simply playing around with the configuration and counting the number of red triangles in a valid case. Here, we formally prove that the problem constraints actually lead to a unique construction of red segments.

Let $B_1, B_2, \cdots, B_9$ be the 9 remaining points. For $1 \leq i \leq 9$, we use x_i to denote the number of red segments connected to B_i. By the given assumption, we have that $0 \leq x_i \leq 9$ and x_i's are all distinct. Without loss if generality, assume that $x_1 < x_2 < \cdots < x_9$.

We claim that $x_9 = 9$. Assume on the contrary that $x_9 \leq 8$. Since $x_{i+1} \geq x_i + 1$ $(1 \leq i \leq 8)$, we can deduce that $x_1 \leq x_9 - 8 \implies x_1 = 0$. As the equalities must be attained, we can further deduce that $x_i = i - 1$ $(\forall 1 \leq i \leq 9)$. Let x_A denote the number of red segments connected to A. By the given assumption, we have that x_A must be odd. Thus, summing the "degrees" of red segments yields

$$\sum_{i=1}^{9} x_i + x_A = \sum_{i=0}^{8} i + x_A = 36 + x_A \equiv 1 \pmod 2$$

This leads to a contradiction, as this sum must be even (each red segment is counted twice in the sum).

Thus, we must have $x_9 = 9$, i.e, B_9 is connected to all the other 9 vertices with one red segment. This implies that $x_i \geq 1$ for $1 \leq i \leq 8$. By applying the property $x_{i+1} \geq x_i + 1$ again, we can deduce that $x_1 \leq x_9 - 8 = 1 \implies x_1 = 1$. As the equality must be attained, we can further deduce that $x_i = i$ $(\forall 1 \leq i \leq 9)$. Given that B_1 is connected to B_9 with one red segment and $x_1 = 1$, we have that there doesn't exist any red segment connecting B_1 and B_8. However, since $x_8 = 8$, we have that B_8 is connected to $B_2, B_3, \cdots, B_9, A$ (all the other points except B_1) with red segments. Similarly, while B_2 is connected to B_9 and B_8 with two red segments while $x_2 = 2$, we can deduce that B_2 is only connected to B_9 and B_8 with red segments. By repeating such analysis, we can obtain the following general conclusion:

For $1 \leq i \leq 4$, B_{9-i} is only connected to $B_{i+1}, B_{i+2}, \cdots, B_9, A$ with $9-i$ red segments, while B_i is only connected to $B_9, B_8 \cdots, B_{10-i}$ with i red segments. This implies that A is only connected to the five vertices B_9, B_8, B_7, B_6, B_5 with five red segments. This shows that this two-colored graph is uniquely determined. Below we will count the number of red triangles in this graph.

Partition the 10 vertices into two subsets: $M = \{B_1, B_2, B_3, B_4\}, N = \{A, B_5, B_6, B_7, B_8, B_9\}$. On the one hand, there does not exist any red segment connecting two vertices in M. On the other hand, any two vertices in N are connected by a red segment. Furthermore, note that for $1 \leq i \leq 4$, $B_i \in M$ is connected to exactly i vertices in N with i red segments. Hence, for $2 \leq i \leq 4$, the number of red triangles with one of its vertices being B_i is given by $\binom{i}{2}$ (when $i = 1$, there's only one red segment connected to B_1, so no red triangle has B_1 as one of its vertices). Note that the number of red triangles in the subgraph formed by N is equal to $\binom{6}{3}$, we can finally conclude that the total number of red triangles in the graph is given by

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{6}{3} = 1 + 3 + 6 + 20 = \boxed{30}.$$

8. Define

$$S = \tan^{-1}(2020) + \sum_{j=0}^{2020} \tan^{-1}(j^2 - j + 1).$$

Then S can be written as $\frac{m\pi}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution: 4045

We see that

$$\begin{aligned}
\sum_{j=0}^{2020} \tan^{-1}(j^2 - j + 1) &= \frac{\pi}{4} + \sum_{j=1}^{2020} \left(\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{j^2 - j + 1}\right)\right) \\
&= \frac{\pi}{4} + \sum_{j=1}^{2020} \left(\frac{\pi}{2} - \tan^{-1}\left(\frac{j - (j-1)}{1 + j(j-1)}\right)\right) \\
&= \frac{\pi}{4} + \sum_{j=1}^{2020} \left(\frac{\pi}{2} - \tan^{-1}(j) + \tan^{-1}(j-1)\right) \\
&= \frac{\pi}{4} + 1010\pi - \tan^{-1}(2020),
\end{aligned}$$

where we utilize the tangent subtraction formula: $\tan^{-1}\left(\frac{j-(j-1)}{1+j(j-1)}\right) = \tan^{-1}(j) - \tan^{-1}(j-1)$. Adding the $\tan^{-1}(2020)$ term, we have that $S = \frac{\pi}{4} + 1010\pi = \frac{4041\pi}{4} \implies \boxed{4045}$.

9. For a positive integer m, let $\varphi(m)$ be the number of positive integers $k \leq m$ such that k and m are relatively prime, and let $\sigma(m)$ be the sum of the positive divisors of m. Find the sum of all even positive integers n such that

$$\frac{n^5\sigma(n) - 2}{\varphi(n)}$$

is an integer.

Solution: 416

If $p^k \mid n$ for some odd prime p with $k \geq 2$ then

$$p \mid \phi(p^k) \mid \phi(n) \implies p \mid n^5\sigma(n) - 2 \implies p \mid 2$$

since $p \mid n^5\sigma(n)$. This is a contradiction.

Now suppose n is not a power of 2. If $2^k \mid n$ with $k \geq 2$, then by the above we can write $n = 2^k p_1 \cdots p_m$ for odd primes $p_1, \ldots, p_m$ with $m \geq 1$. $\varphi(n) = 2^{k-1}\prod_{i=1}^{m}(p_i - 1)$, so

$$v_2(\varphi(n)) \geq k - 1 + m \geq 1 + 1 = 2$$

as all the p_i are odd. This implies that $4 \mid n^5\sigma(n) - 2$ which is a contradiction as

$$2^k \mid n, k \geq 2 \implies n \equiv 0 \pmod 4 \implies n^5\sigma(n) - 2 \equiv 2 \pmod 4$$

Thus $n = 2p_1 \cdots p_m$ for odd primes $p_1, \ldots, p_m$ with $m \geq 1$. If $m \geq 2$, then as all the p_i are odd,

$$v_2(\varphi(n)) \geq m \geq 2$$

which again is a contradiction:

$$n \equiv 0 \pmod 2 \implies n^5 \equiv 0 \pmod 4 \implies n^5\sigma(n) - 2 \equiv 2 \pmod 4$$

This means that if n is not a power of 2, then $n = 2p$ for an odd prime p, so $\sigma(n) = 1+2+p+2p = 3p+3$. The condition means

$$p - 1 \mid 32p^5(3p+3) - 2 \implies p - 1 | 190$$

since the remainder of $32p^5(3p+3) - 2$ upon division by $p-1$ is $32 \cdot 1 \cdot (6) - 2 = 190$ by the Polynomial Remainder Theorem. $190 = 2 \cdot 5 \cdot 19$ and $p - 1$ must be even, so $p = 3, 11, 191$ since 39 is not a prime. By the above reasons, all these solutions work; we have $n = 6, 22, 382$ as possible solutions.

Otherwise suppose n is a power of two, so write $n = 2^k$. Since n is even, $k \geq 1$. We also have $\sigma(n) = 1 + 2 + \cdots + 2^k = 2^{k+1} - 1$. Plugging into the condition, $2^{k-1} \mid 2^{5k}(2^{k+1} - 1) - 2$. If $k \geq 3$, $2^{k-1} \equiv 0 \pmod 4, 2^{5k}(2^{k+1} - 1) - 2 \equiv 2 \pmod 4$ which is a contradiction. However, we can check that $k = 1, 2$ both work. This yields the solutions $n = 2, 4$.

So the answer is $2 + 4 + 6 + 22 + 382 = \boxed{416}$.

10. A research facility has 60 rooms, numbered $1, 2, \ldots 60$, arranged in a circle. The entrance is in room 1 and the exit is in room 60, and there are no other ways in and out of the facility. Each room, except for room 60, has a teleporter equipped with an integer instruction $1 \leq i < 60$ such that it teleports a passenger exactly i rooms clockwise.

On Monday, a researcher generates a random permutation of $1, 2, \ldots, 60$ such that 1 is the first integer in the permutation and 60 is the last. Then, she configures the teleporters in the facility such that the rooms will be visited in the order of the permutation.

On Tuesday, however, a cyber criminal hacks into a randomly chosen teleporter, and he reconfigures its instruction by choosing a random integer $1 \leq j' < 60$ such that the hacked teleporter now teleports a passenger exactly j' rooms clockwise (note that it is possible, albeit unlikely, that the hacked teleporter's instruction remains unchanged from Monday). This is a problem, since it is possible for the researcher, if she were to enter the facility, to be trapped in an endless "cycle" of rooms.

The probability that the researcher will be unable to exit the facility after entering in room 1 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution: 88

Let $a_1, a_2, \ldots, a_{60}$ be the unique permutation of $1, 2, \ldots, 60$ such that a_i is the ith room visited. Obviously, $a_1 = 1, a_{60} = 60$. Assuming no glitch, the path of rooms visited by the researcher is clearly

$$a_1 \to a_2 \to \cdots \to a_{59} \to a_{60}.$$

If the teleporter in room a_1 is hacked, then the path $a_1 \to a_2$ becomes $a_1 \to a_{j'}$ for some random $j' \in \{2, 3, \ldots, 60\}$. Thus, the researcher will always be able to eventually reach a_{60}. This case occurs with a $\frac{1}{59}$ probability, and has a 0 probability of the researching failing to exit the facility.

If the teleporter in room a_i is hacked, where $i \in \{2, 3, \ldots, 59\}$ then the path $a_i \to a_{i+1}$ becomes $a_i \to a_{j'}$ for some random $j' \in \{1, 2, \ldots 60\}, j' \neq i$. We see that the researcher can exit the facility if and only if $j' \geq i + 1$. Otherwise, the "farthest" room the researcher can reach will be a_i, and the path $a_i \to a_{j'}$ will send the researcher "back", causing them to be stuck in an infinite loop. This case occurs with a $\frac{1}{59}$ probability (for each i), and has a $\frac{i-1}{59}$ chance of the researcher failing to exit the facility.

Hence, the total probability the researcher is unable to exit the facility is given by

$$\frac{1}{59} \cdot \sum_{i=2}^{59} \frac{i-1}{59} = \frac{1}{59} \cdot \frac{58 \cdot 59}{2 \cdot 59} = \frac{29}{59}.$$

The answer is thereby $\boxed{88}$.

11. Let $n \geq 3$ be a positive integer. Suppose that Γ is a unit circle passing through a point A. A regular 3-gon, regular 4-gon, $\ldots$, regular n-gon are all inscribed inside Γ such that A is a common vertex of all these regular polygons. Let Q be a point on Γ such that Q is a vertex of the regular n-gon, but Q is not a vertex of any of the other regular polygons. Let $\mathcal{S}_n$ be the set of all such points Q. Find the number of integers $3 \leq n \leq 100$ such that

$$\prod_{Q \in \mathcal{S}_n} |AQ| \leq 2.$$

Solution: 68

If we label the other points of the regular n-gon $Q_1, Q_2, \ldots, Q_{n-1}$, we see that the regular n-gon $AQ_1Q_2 \ldots Q_{n-1}$ only shares the vertex A with a regular k-gon if k is relatively prime to n. Thus, it is possible for some Q_j to lie on *some* regular k-gon containing point A (where $k < n$) if and only if k and n share some common divisor greater than 1. Suppose that the greatest common divisor of j and n is greater than 1. Then, if we let $k = \frac{n}{d}$, where d is any divisor of $\gcd(j, n)$, we see that the regular k-gon passes through point Q_j. Hence, the set $\mathcal{S}_n$ of all points Q_j that only lie on the regular n-gon are those that satisfy $\gcd(j, n) = 1$.

There is one exception to this conclusion, and it is the case $n = 4$. This is because there is no "regular 2-gon". We will address this case at the end of our solution.

Our desired product is

$$\prod_{1 \leq i \leq n, \gcd(j,n)=1} |AQ_j| \leq 2.$$

If we let $w = e^{\frac{2\pi i}{n}}$ (a primitive nth root of unity), observe that $|AQ_j| = |1 - w^j|$. Furthermore, we know that the polynomial with the primitive nth roots of unity is the cyclotomic polynomial $\Phi_n(z)$.

Hence, we see that

$$\Phi_n(z) = \prod_{1\le j\le n, \gcd(j,n)=1} (z - w^j)$$
$$\implies \prod_{1\le j\le n, \gcd(j,n)=1} |AQ_j| = \prod_{1\le j\le n, \gcd(j,n)=1} |1 - w^j| = |\Phi_n(1)|,$$

since each of the terms w^j describes a primitive nth root of unity. Thus, we want to find all n such that $|\Phi_n(1)| = 1$. Let p be a prime number and m be a positive integer. We claim that

$$\Phi_n(1) = \begin{cases} p & n = p^m \\ 1 & \text{otherwise} \end{cases}.$$

We prove the first case (n is a prime power) by strong induction on m.

For the base case $m = 1$, note that $\Phi_n(z) = 1 + z + \cdots + z^{n-1}$, so $\Phi_n(1) = n$.

For the induction step $m > 1$, assume the statement is true for all $1 \le m_0 < m$. As a well-known property of cyclotomic polynomials, we have that $z^n - 1 = \prod_{d|n} \Phi_d(z)$, so $f(z) = 1 + z + \cdots + z^{n-1} = \prod_{j=1}^{m} \Phi_{p^j}(z)$. Observe that $f(1) = p^m$ on the left hand side. On the right hand side, we have that $\Phi_{p^j}(1) = p$ term for each $j < m$, so the equation simplifies to $f(1) = p^m = p^{m-1}\Phi_n(1) \implies \Phi_n(1) = p$.

Now suppose that n is the product of $s \ge 2$ distinct primes: $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$. Once again, we see that $f(z) = 1 + z + \cdots + z^{n-1} = \prod_{d|n, d\ne 1} \Phi_d(z)$. Let $\mathcal{D}$ be the nonempty set of all divisors of n that are a product of > 1 distinct primes. We observe that

$$\prod_{d\in\mathcal{D}} \Phi_d(z) = \frac{\prod_{d|n,d\ne1} \Phi_d(z)}{\prod_{j=1}^{s} \prod_{l=1}^{\alpha_j} \Phi_{p_j^l}(z)}$$
$$\implies \prod_{d\in\mathcal{D}} \Phi_d(1) = \frac{\prod_{d|n,d\ne1} \Phi_d(1)}{\prod_{j=1}^{s} \prod_{l=1}^{\alpha_j} \Phi_{p_j^l}(1)} = \frac{n}{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}} = 1.$$

Cyclotomic polynomials are monic, have integer coefficients, and do not have any real roots $n > 2$, so the value of $\Phi_n(z)$ is a positive integer for all integers z. However, $\prod_{d\in\mathcal{D}} \Phi_d(1) = 1$, so we see that $\phi_d(1) = 1$ for all $d \in \mathcal{D}$, including n.

Finally, for the case $n = 4$, there is a square $AQ_1Q_2Q_3$ which does not share any vertices with the equilateral triangle other than A, so the desired product is $|AQ_1||AQ_2||AQ_3| = \sqrt{2} \cdot 2 \cdot \sqrt{2} = 4 > 2$.

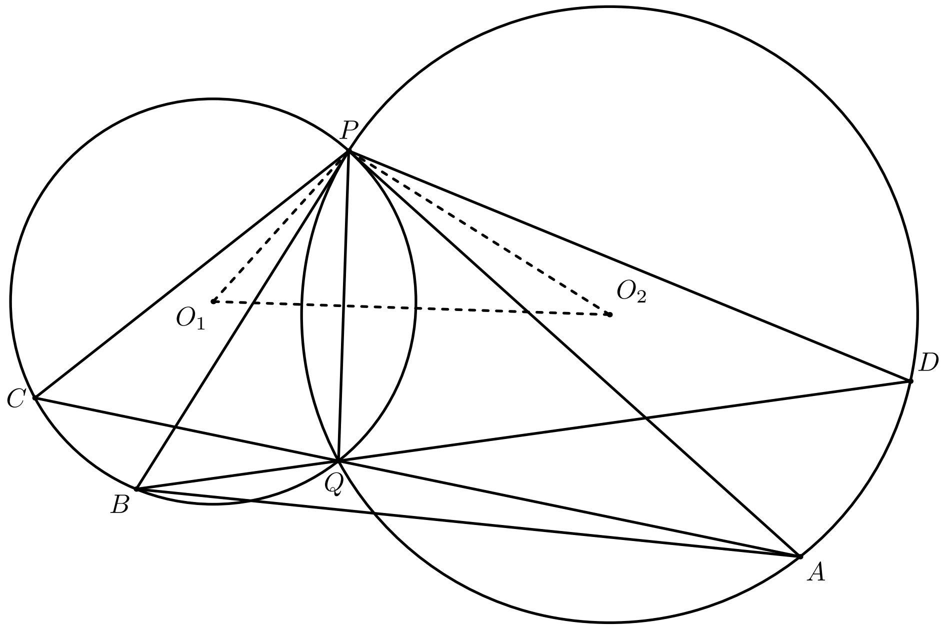
Thus, the the number of desired integers is the 98 integers from 3 to 100 inclusive minus the primes and prime powers excluding $8, 16, 32, 64$, the prime powers of 2 except for 4. We have a total of 34 primes and prime powers from 3 to 100, so the answer is $98 - 34 + 4 = \boxed{68}$.

12. Let Ω_1 and Ω_2 be two circles intersecting at distinct points P and Q. The line tangent to Ω_1 at P passes through Ω_2 at a second point A, and the line tangent to Ω_2 at P passes through Ω_1 at a second point B. Ray AQ intersects Ω_1 at a second point C, and ray BQ intersects Ω_2 at a second point D. Suppose that $\angle CPD > \angle APB$ (measuring both angles as the non-reflex angle) and that

$$\frac{\text{Area}(CPD)}{PA \cdot PB} = \frac{1}{4}.$$

Find the sum of all possible measures of $\angle APB$ in degrees.

Solution: 130



Let O_1, O_2 be the respective centers of Ω_1, Ω_2. We claim that $\triangle CPA \sim \triangle O_1PO_2 \sim \triangle BPD$. To see why, note that $\angle PO_1O_2$ subtends an arc equal to $\frac{1}{2}$ of $\widehat{PQ}$ in Ω_1, and $\angle PO_2O_1$ subtends an arc equal to $\frac{1}{2}$ of $\widehat{PQ}$ in Ω_2. Since O_1, O_2 are circle centers, we see that $\angle PCA \cong \angle PBD \cong \angle PO_1O_2$ and $\angle PAC \cong \angle PDB \cong \angle PO_2O_1$, so the respective triangles are similar by AA.

Since PB is tangent to Ω_2, we see that $m\angle BPO_2 = 90°$. Similarly, $m\angle APO_1 = 90°$. Thus, we can let $m\angle O_1PB = m\angle O_2PA = \theta$ for some real number θ. Hence, $m\angle BPA = 90° - \theta, m\angle O_1PO_2 = m\angle CPA = m\angle BPD = 90° + \theta$, so we also conclude that $m\angle CPO_1 = m\angle DPO_2 = \theta$. Since PO_1, PO_2, respective radii of Ω_1, Ω_2, bisect respective angles $\angle CPB$ and $\angle DPA$, we conclude that $PC = PB$ and $PD = PA$. Furthermore, $m\angle CPD = 90° + 3\theta$. Since $\angle CPD > \angle APB$, we must have that $\theta > 0$. Finally, we observe that

$$\frac{\text{Area}(CPD)}{PA \cdot PB} = \frac{1}{4} = \frac{\frac{1}{2} \cdot PC \cdot PB \cdot \sin(CPD)}{PC \cdot PB} = \frac{1}{2} \cdot \sin(CPD),$$

so we want $\sin(CPD) = \sin(90° + 3\theta) = \frac{1}{2}$. Since θ can be at most $90°$ (as $m\angle O_1PO_2 \leq 180°$, we have that $90° + 3\theta = 150°, 210°, 330°$. This gives values of $\theta : 20°, 40°, 80°$. The possible values of $\angle APD$ are therefore $90 - \theta$; in degrees, that is $\{10, 50, 70\}$. Also, we can easily check that the non-reflex $\angle CPD$ (which is $150°$ and $30°$) is clearly larger than the respective $\angle APD$. Hence, the answer is $\boxed{130}$.

13. Let a, b, c, d be reals such that $a \geq b \geq c \geq d$ and

$$\begin{aligned}&(a-b)^3 + (b-c)^3 + (c-d)^3 - 2(d-a)^3 \\ &- 12(a-b)^2 - 12(b-c)^2 - 12(c-d)^2 + 12(d-a)^2 \\ &- 2020(a-b)(b-c)(c-d)(d-a) = 1536.\end{aligned}$$

Find the minimum possible value of $d - a$.

Solution: -8

Substitute nonnegative real numbers $x = a - b, y = b - c, z = c - d$ to obtain

$$x^3 + y^3 + z^3 + 2(x+y+z)^3 + 24(xy + yz + xz) + 2020(x+y+z)xyz = 1536.$$

Substitute $p = x + y + z$, $q = xy + yz + xz$, $r = xyz$ to obtain

$$p^3 - 3pq + 3r + 24q + 2p^3 + 2020pr = 1536.$$

If we let $p = 8$, the equation becomes $(3 + 2020 \cdot 8)r + 1536 = 1536$, which is possible if at least one of $x, y, z = 0$. If $p > 8$, then the equation is

$$3q(8 - p) + 3r + 2020pr = 1536 - 3p^3$$

and we claim this is impossible. To see why, we note that $q < p^2 \implies q < p^2 + 8p + 64 \implies 3q(8 - p) > 1536 - 3p^3$; however, $3r + 2020pr$ is nonnegative, so the above equation cannot hold.

Hence, the maximum of $p = x+y+z$ is 8. However, $x+y+z = a-d$, implying that the minimum possible value of $d - a$ is the negative, $\boxed{-8}$.

14. Let a be a positive real number. Collinear points Z_1, Z_2, Z_3, Z_4 (in that order) are plotted on the (x, y) Cartesian plane. Suppose that the graph of the equation

$$x^2 + (y + a)^2 + x^2 + (y - a)^2 = 4a^2 + \sqrt{(x^2 + (y + a)^2)(x^2 + (y - a)^2)}$$

passes through points Z_1 and Z_4, and the graph of the equation

$$x^2 + (y + a)^2 + x^2 + (y - a)^2 = 4a^2 - \sqrt{(x^2 + (y + a)^2)(x^2 + (y - a)^2)}$$

passes through points Z_2 and Z_3. If $Z_1Z_2 = 5$, $Z_2Z_3 = 1$, and $Z_3Z_4 = 3$, then a^2 can be written as $\frac{m+n\sqrt{p}}{q}$, where m, n, p, and q are positive integers, m, n, and q are relatively prime, and p is squarefree. Find $m + n + p + q$.

Solution: 144

Let $P = (a, 0)$, $Q = (-a, 0)$. By the Law of Cosines, we can see that

$$x^2 + (y + a)^2 + x^2 + (y - a)^2 = 4a^2 + \sqrt{(x^2 + (y + a)^2)(x^2 + (y - a)^2)}$$

is the set of all points $\mathcal{S}_1$ such that $\forall R_1 \in \mathcal{S}_1$, $\angle PR_1Q = 60°$ or $R_1 = P, Q$. Likewise, we can see that

$$x^2 + (y + a)^2 + x^2 + (y - a)^2 = 4a^2 - \sqrt{(x^2 + (y + a)^2)(x^2 + (y - a)^2)}$$

is the set of all points $\mathcal{S}_2$ such that $\forall R_2 \in \mathcal{S}_2$, $\angle PR_2Q = 120°$ or $R_2 = P, Q$. Hence, these two sets collectively describe the circles ω_1, ω_2 given by the following two equations:

$$\omega_1 : \left(x + \frac{r}{2}\right)^2 + y^2 = r^2,\ \omega_2 : \left(x - \frac{r}{2}\right)^2 + y^2 = r^2,$$

where $r = \frac{2a}{\sqrt{3}}$.

WLOG let Z_1, Z_2, Z_3, Z_4 have increasing x coordinates. Then we have that $Z_1, Z_3 \in \omega_1$, $Z_2, Z_4 \in \omega_2$. Let O_1, O_2 be the respective centers of ω_1, ω_2. Further let H_1, H_2 be the respective feet of the perpendiculars onto Z_1Z_4 from O_1, O_2. Then, we see that $Z_1H_1 = \frac{Z_1Z_3}{2} = 3$ and $H_2Z_4 = \frac{Z_2Z_4}{2} = 2$, so $H_1H_2 = 4$, $O_1H_1 = \sqrt{r^2 - 9}$, and $O_2H_2 = \sqrt{r^2 - 4}$. Also, ω_1 and ω_2 pass through each others' centers, so $O_1O_2 = r$.

Therefore, by the Pythagorean Theorem on right trapezoid $O_1O_2H_2H_1$,

$$4^2 + \left(\sqrt{r^2 - 4} - \sqrt{r^2 - 9}\right)^2 = r^2 \implies r^2 = \frac{29 \pm 2\sqrt{109}}{3},$$

of which only the positive solution yields real values of $\sqrt{r^2 - 4}, \sqrt{r^2 - 9}$. Hence, $a^2 = \frac{29+2\sqrt{109}}{4}$, and the answer is $\boxed{144}$.

15. For an integer $n \geq 2$, let G_n be an $n \times n$ grid of unit cells. A subset of cells $H \subseteq G_n$ is considered *quasi-complete* if and only if each row of G_n has at least one cell in H and each column of G_n has at least one cell in H. A subset of cells $K \subseteq G_n$ is considered *quasi-perfect* if and only if there is a proper subset $L \subset K$ such that $|L| = n$ and no two elements in L are in the same row or column. Let $\vartheta(n)$ be the smallest positive integer such that every quasi-complete subset $H \subseteq G_n$ with $|H| \geq \vartheta(n)$ is also quasi-perfect. Moreover, let $\varrho(n)$ be the number of quasi-complete subsets $H \subseteq G_n$ with $|H| = \vartheta(n) - 1$ such that H is not quasi-perfect. Compute $\vartheta(20) + \varrho(20)$.

Solution: 7963

We claim that $\vartheta(n) = n^2 - 2n + 3, \varrho(n) = n^2(n-1)$. For $n = 2$, this is easy to check manually, so now suppose $n \geq 3$.

Consider the subset $K \subseteq G_n$ containing every entry of the first $n-2$ columns of G_n and the first entry of the last 2 columns of G_n, where we read the columns of G_n from left to right. K is clearly quasi-complete but it is not quasi-perfect, so it follows that $\vartheta(n) \geq n^2 - 2n + 3$.

Now we show that every quasi-complete subset $H \subset G_n$ with $|H| = \vartheta(n) = n^2 - 2n + 3$ is also quasi-perfect. Consider the elements in H in each of the rows of G_n as a family of n subsets of $\{1, 2, \ldots, n\}$, H is quasi-perfect iff there is a transversal through these sets. To this end, take the permutation $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ and the nondecreasing sequence $a_1, a_2, \ldots, a_n$ such that a_i is the number of elements in H that are in row $\sigma(i)$ of G_n. H is quasi-complete, so $a_1 \geq 1$. By Hall's Marriage Theorem, it is *possible* for H to not be quasi-perfect only if the "violation" $a_i < i$ occurs for some $2 \leq i \leq n-1$ (note: we do not worry about a_n since we assume that H is quasi-complete).

Let $f(i) = i(i-1) + n(n-i)$, this is the maximum number of elements H can have for some "violation" $a_i < i$. The inequality $f(i) \geq |H| = n^2 - 2n + 3$ must be satisfied. $f(i)$ is convex, so the absolute maxima of $f(i)$ on $[2, n-1]$ occur at the extremes $i = 2, n-1$, where $f(i) = n^2 - 2n + 2$. The maxima do not satisfy the above inequality, so H is always quasi-perfect.

Now, we compute the number of quasi-complete subsets $H \subset G_n, |H| = \vartheta(n) - 1 = n^2 - 2n + 2$ that are not quasi-perfect. Define $a_1, a_2, \ldots, a_n$ and $f(n)$ as before. Since $|H| = n^2 - 2n + 3$, we must either have $a_n = n - 1$ or $a_n = n$.

If $a_n = n - 1$, then the inequality $f(i) - 1 \geq |H| = n^2 - 2n + 2$ must be satisfied for some "violation" $a_i < i$. By the previous argument, this inequality is never satisfied. If $a_n = n$, then the required inequality is $f(i) \geq |H| = n^2 - 2n + 2$, which is satisfied for the cases $i = 2, n-1$ (it is an equality). In fact, the sequence $a_1, a_2, \ldots, a_n$ is either $1, 1, \underbrace{n, n, \ldots, n}_{n-2 \text{ numbers}}$ or $\underbrace{n-2, n-2, \ldots, n-2}_{n-1 \text{ numbers}}, n$.

On the one hand, if $a_1, a_2, \ldots, a_n = 1, 1, n, n, \ldots, n$, then observe by Hall's Marriage Theorem that the two rows with 1 element in H share the same column. There are $\binom{n}{2}$ ways to choose the two rows with 1 element, and n ways to choose the shared column, yielding $\frac{n^2(n-1)}{2}$ non quasi-perfect subsets H. On the other hand, if $a_1, a_2, \ldots, a_n = n-2, n-2, \ldots, n-2, n$, we can visualize this as a $90°$ "rotation" of the $n \times n$ grid from the previous case. Thus, this case also yields $\frac{n^2(n-1)}{2}$ non quasi-perfect subsets H. Hence, $\varrho(n) = n^2(n-1)$.

The requested sum is $\vartheta(20) + \varrho(20) = 363 + 7600 = \boxed{7963}$.

Remark: for all $n + 1 \geq k \geq \vartheta(n) - 1$, consider a subset $H \subseteq K \subset G_n, |H| = k, |K| = \vartheta(n) - 1$. If K is quasi-complete but not quasi-perfect, we may also carefully "construct" H such that it is quasi-complete but not quasi-perfect. If $L \subset G_n, |L| \leq n$, then L cannot be quasi-perfect: it has no proper quasi-complete subset.