Chapter 2

Set Theory

2.1 Sets

The most basic object in Mathematics is called a *set*. As rudimentary as it is, the exact, formal definition of a set is highly complex. For our purposes, we will simply define a **set** as a collection of objects that is well-defined. That is, it is possible to determine if an object is to be included in the set or not.

Frequently a set is denoted by a capital letter, like S. Objects in the set collection are known as **elements** of S. If x is one of these elements, then we write that $x \in S$. Similarly, if an object x is not a part of the set, then we write $x \notin S$.

2.1.1 Examples of Sets and their Elements

The most basic set is the collection of no objects. This set, known as the **empty set** or **null set**, is denoted by $\varnothing$ or $\{\}$, to indicate that it contains no elements.

In Mathematics, the most frequently encountered sets are various collections of types of real numbers. The below such sets are of key importance.

- The set of **natural numbers**, denoted by $\mathbb{N}$, is the set of all non-negative whole numbers. Thus, we can list off a few of the elements by writing

  $$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

  Note that some mathematicians have the natural numbers defined as the set of all positive whole numbers.

- The set of **integers**, denoted by $\mathbb{Z}$, is the set of all whole numbers. Thus, we can list them off as

  $$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, 3, \ldots\}.$$

- The set of all **real numbers** is denoted by $\mathbb{R}$.

It is important to note that sets are *unordered* objects. Thus, the set $\{a, b, c\}$ and the set $\{b, c, a\}$ are considered the same because they contain the same elements, even if they are listed in different orders.

2.1.2 Set-Builder Notation

If we wish to efficiently describe sets of objects by some defining characteristic or by restricting a larger set via some conditions, then we may use **set-builder notation**. A set S described in set-builder notation has the form

$$S = \{x \mid p(x)\},$$

where $p(x)$ is some statement about x. This set should be read as "S is the set of all x such that $p(x)$ is true." Below are some examples of sets described via set-builder notation.

- We can describe the natural numbers $\mathbb{N}$ by placing a restriction on the set of integers:
$$\mathbb{N} = \{x \in \mathbb{Z} \mid x \geq 0\}.$$
- We can describe the set of *rational numbers*, denoted by $\mathbb{Q}$, in set-builder notation as
$$\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\},$$
where we have the understanding that two rational numbers $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ are equal if $p_1 q_2 = p_2 q_1$. Thus, the set of all rational numbers is equal to the set of all fractions where the numerator and denominator are integers with the denominator not being 0.
- The set of complex numbers, denoted by $\mathbb{C}$, can be written in set-builder notation as
$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}.$$
- We can describe specific sets using set-builder notation as well. For instance, the set
$$S = \{n \in \mathbb{Z} \mid -2 \leq n < 5\}$$
is just the set $\{-2, -1, 0, 1, 2, 3, 4\}$.
- The set
$$S = \{x \in \mathbb{R} \mid x^2 - 1 = 0\}$$
is simply the two element set $\{-1, 1\}$.
- The set
$$\{x \in \mathbb{R} \mid -2 \leq x\}$$
can be written in interval notation as the closed ray $[-2, \infty)$.

2.1.3 Subsets

If we compare the two sets $\mathbb{N}$ and $\mathbb{Z}$, it is clear that every single natural number in $\mathbb{N}$ is also an integer in $\mathbb{Z}$. As such, we can think of the set $\mathbb{N}$ as being contained inside of $\mathbb{Z}$. In fact, we would say that $\mathbb{N}$ is a *subset* of $\mathbb{Z}$.

More formally, we have that a set A is a **subset** of a set S if whenever $x \in A$, then $x \in S$. Below are some examples of subsets.

- The empty set $\varnothing$ is a subset of every set S. Notice that it is true that if $x \in \varnothing$, then $x \in S$ because there are no $x \in \varnothing$ for which we need to check this condition. Thus,
$$\varnothing \subset S$$
for every set S.

- The integers $\mathbb{Z}$ are a subset of the rationals $\mathbb{Q}$ since, if $n \in \mathbb{Z}$, then it can be written as $\frac{n}{1}$ as is thus an element of $\mathbb{Q}$. Thus,

$$\mathbb{Z} \subset \mathbb{Q}.$$

- The real numbers $\mathbb{R}$ are a subset of the complex numbers $\mathbb{C}$ since, if $a \in \mathbb{R}$, then it can be written as $a + 0i$, which is an element of $\mathbb{C}$. Thus,

$$\mathbb{R} \subset \mathbb{C}.$$

2.1.4 Proofs Involving Subsets

Since the definition of $A \subset S$ says that if $x \in A$, then $x \in S$, then we can prove that A is indeed a subset of S by assuming that we have an arbitrary element $x \in A$ and then show that $x \in S$ as well. In the above examples, this is precisely what we did. Below, we will see a slightly more complicated example of a proof of subset inclusion.

Proposition. Let $S = (-3, 5)$ and $T = (-6, 5]$. Then, $S \subset T$.

Discussion.

`What we know:` We can re-write the interval notation sets in set-builder notation as

$$(-3, 5) = \{x \in \mathbb{R} \mid -3 < x < 5\} \text{ and}$$
$$(-6, 5] = \{x \in \mathbb{R} \mid -6 < x \leq 5\}.$$

`What we want:` $S \subset T$. Thus, we will be taking an arbitrary element $x \in S$ and then, using the definition of S and T, show that $x \in T$ as well.

Proof. Assume $x \in S$. Then, x satisfies $-3 < x < 5$. We will show that $x \in T$ by showing that x satisfies $-6 < x \leq 5$. Since $-3 < x$, we also know that $-6 < -3 < x$ and thus $-6 < x$. Furthermore, since $x < 5$ and $5 \leq 5$, then $x \leq 5$. So, x satisfies $-6 < x \leq 5$, as desired. So, $x \in T$ and $S \subset T$.

□

We can also use the definition of a subset to show a fact that seems obvious: If A is a subset of B and B is a subset of C, then A is a subset of C. Notice that, in this case, we do not have specific sets, so we can only rely on the definition of a subset to prove the above statement.

Proposition. Let A, B, and C be sets. If $A \subset B$ and $B \subset C$, then $A \subset C$.

Discussion.

`What we know:` We know that

- $A \subset B$; thus, whenever we know that $x \in A$, we can conclude that $x \in B$.
- $B \subset C$; thus, whenever we know that $x \in B$, we can conclude that $x \in C$.

`What we want:` We want to show that $A \subset C$. Thus, we are going to assume that $x \in A$ and will eventually conclude that $x \in C$.

Proof. Let $x \in A$. Since $A \subset B$ and $x \in A$, we can conclude that $x \in B$. Since $B \subset C$ and $x \in B$, we can conclude that $x \in C$. Thus, $A \subset C$.

□

2.1.5 Unions and Intersections

If we have two sets S and T, then we can form new sets that are combinations of the elements of S and T. First we defined the **union** of our sets as

$$S \cup T = \{x \,|\, x \in S \text{ or } x \in T\}.$$

Thus, $S \cup T$ is the set that contains all the elements in S as well as all the elements in T. We remember, of course, that, the "or" in this definition means that if $x \in S$ and also $x \in T$, then it is also included as an element of the union $S \cup T$.

Notice that, since every element $x \in S$ is also included in $S \cup T$, then we can conclude that

$$S \subset S \cup T.$$

Similarly, we also know that

$$T \subset S \cup T.$$

If instead of using "or", we use the conjunction "and", then we arrive at the definition of an **intersection** of two sets:

$$S \cap T = \{x \,|\, x \in S \text{ and } x \in T\}.$$

Thus, for an element to be in the intersection of two sets, it has to meet the two conditions that it be an element of S and also an element of T. Notice that if $x \in S \cap T$, then, by definition $x \in S$ and also $x \in T$. Thus, we can conclude that

$$S \cap T \subset S \text{ and } S \cap T \subset T.$$

Two sets S and T are called **disjoint** if they share no elements in common. In other words, S and T are disjoint when $S \cap T = \varnothing$.

Below are some examples of unions and intersections.

- Let D_n be the set of all natural numbers that divide the natural number n. Thus, for example,

  $$D_{20} = \{1, 2, 4, 5, 10, 20\} \text{ and } D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}.$$

  Then, $D_{20} \cap D_{24} = \{1, 2, 4\}$. Notice that the largest number in this intersection is exactly the *greatest common divisor* of 20 and 24.

- Let M_n be the set of all positive multiples of the natural number n. Thus for example,

  $$M_{12} = \{12, 24, 36, 48, 60, 72, 84, 96, 108, \ldots\}$$

  $$M_{18} = \{18, 36, 54, 72, 90, 108, \ldots\}.$$

  The intersection of these sets is given by $M_{12} \cap M_{18} = \{36, 72, 108, \ldots\}$. Notice that the smallest of these numbers is exactly the *least common multiple* of 12 and 18.

- Consider the sets S and T given by

  $$S = \{x \in \mathbb{R} \,|\, \sin x = 0\}$$

  $$T = \{x \in \mathbb{R} \,|\, \cos x = 0\}.$$

  The set S is simply the *roots* of the function $\sin x$ and is thus equal to

  $$S = \{\pi k \,|\, k \in \mathbb{Z}\} = \{\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots\};$$

in other words, it is the same as all integers multiples of π. Similarly, T is the roots of $\cos x$, which is equal to $\frac{\pi}{2}$ plus integers multiples of π and can thus be written as

$$T = \left\{ \frac{\pi}{2} + \pi k \,\middle|\, k \in \mathbb{Z} \right\} = \{\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\}.$$

So, the union of S and T is equal to

$$S \cup T = \{..., -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots\}$$

and can thus be re-written as

$$S \cup T = \left\{ \frac{k\pi}{2} \,\middle|\, k \in \mathbb{Z} \right\}.$$

Notice that since S gave the roots of $\sin x$ and T gave the roots of $\cos x$, that the union $S \cup T$ is exactly the roots of the product $\sin x \cos x$. Furthermore, note that the intersection $S \cap T = \varnothing$ since the two sets of roots are *disjoint* (i.e., they share no common elements).

2.1.6 Complements of Sets

It is just as important to know which objects are elements of a set as to know which objects are not elements of a set. If $A \subset S$, then we can define the **complement** of A in S as the set of all element in S that are not in A:

$$\overline{A} = \{x \in S \,|\, x \notin A\}.$$

In other texts, complements may also be written as A^c, but we will stick to the $\overline{A}$ convention.

Notice that, when taking the complement of A, it is important that the larger set S is also understood. For example,

- If $A = \{0, 1\}$ is the set of two elements 0 and 1, and we are taking its complement in $\mathbb{N}$, then
$$\overline{A} = \{2, 3, 4, \ldots\}.$$
- If $A = \{0, 1\}$, but now, we take its complement in the set of all reals $\mathbb{R}$, then
$$\overline{A} = (-\infty, 0) \cup (0, 1) \cup (1, \infty).$$

Many important examples of complements of sets exist. For example, if we consider $\mathbb{Q} \subset \mathbb{R}$, then the complement of $\mathbb{Q}$ in $\mathbb{R}$ is given by $\overline{\mathbb{Q}}$ and is the set of *irrational numbers*.

We will now prove a statement about the complement of sets that seems believable. Consider A and B, which are subsets of a set S; furthermore, consider their complements in S. We will prove if $A \subset B$, then $\overline{B} \subset \overline{A}$. Since we will be asked to prove a statement about complements, we will be dealing with negative statements (for example, $x \notin A$). As such, we will use the technique of *proof by contradiction*, where we will show something is true by assuming the negation of what we want and arriving at a contradiction. Since we arrived at this contradiction, our initial assumption (that the negation of what we want is true) must be false and thus what we want is true.

Proposition. Let $A, B \subset S$. If $A \subset B$, then $\overline{B} \subset \overline{A}$.

Discussion. In this proof, we will be using the technique of *proof by contradiction*. In particular, when we prove that $\overline{B} \subset \overline{A}$, we will want to conclude that $x \in \overline{A}$, which is equivalent to $x \notin A$. Thus, we will assume that $x \in A$ and arrive at a contradiction.

`What we know:` $A \subset B$: if we ever know that $x \in A$, then we can conclude that $x \in B$.

`What we want:` $\overline{B} \subset \overline{A}$: We will assume that $x \in \overline{B}$ and our job is to conclude that $x \in \overline{A}$.

`What we'll do:` Since we wish to show that $\overline{B} \subset \overline{A}$, we will assume that $x \in \overline{B}$, which is equivalent to $x \notin B$. Our job is to show that $x \in \overline{A}$, which is equivalent to $x \notin A$. Since we wish to show that $x \notin A$, we will use *proof by contradiction*. So, we will assume $x \in A$ and arrive at a contradiction. When we reach this contradiction, our initial assumption that $x \in A$ will be false and thus $x \notin A$ will be true.

Proof. Assume that $x \in \overline{B}$. We will show that $x \in \overline{A}$ by showing that $x \notin A$. Assume, to the contrary, that $x \in A$. Since $x \in \overline{B}$, then $x \notin B$. Since $x \in A$ and since $A \subset B$, we conclude that $x \in B$. However, we also know that $x \notin B$, a contradiction. Thus, our initial assumption that $x \in A$ is false and thus $x \notin A$ is true. So, $x \in \overline{A}$, and we conclude that $\overline{B} \subset \overline{A}$.

□

It is now tempting to make various claims about complements of sets. For example, it seems very obvious that if $A \subset S$, then

$$A \cap \overline{A} = \varnothing \text{ and } A \cup \overline{A} = S.$$

However, we have never formally defined what it means for two sets to be equal to each other. In the next section, we will investigate the formal definition of set equality.

2.2 Definition of Set Equality

It seems reasonable to define the equality of two sets by saying something along the lines of "S and T are equal if they contain exactly the same elements." In proofs, though, this does not give us much of a methodology by which to show that two sets are equal. Thus, to formalize the definition of set equality, we use the concept of subsets. Notice that every set is a subset of itself. Thus, if S and T are to be **equal as sets**, then $S \subset T$ and $T \subset S$. Thus, in a proof, to show that $S = T$, we must show two subset inclusions: $S \subset T$ and $T \subset S$.

Below is an example of a proof that the two sets regarding the roots of $\sin x$ and $\cos x$ are indeed equal.

Proposition. Consider the sets

$$S = \{x \in \mathbb{R} \mid \sin x = 0\} \text{ and } T = \{x \in \mathbb{R} \mid \cos x = 0\},$$

along with the set

$$R = \{x \in \mathbb{R} \mid \sin x \cos x = 0\}.$$

Then, $S \cup T = R$.

Discussion.

`What we want:` $S \cup T = R$. Thus, we need to show two subset inclusions: $S \cup T \subset R$ and $R \subset S \cup T$.

`What we'll do:` For the first subset inclusion $S \cup T \subset R$, we will let $x \in S \cup T$. Thus, $x \in S$ or $x \in T$. Anytime that we know an "or" to be true, we will want to use cases. In this case, we have two cases: $x \in S$ or $x \in T$. For each case, we must conclude that $x \in R$.

For the other subset inclusion $R \subset S \cup T$, we will assume that $x \in R$. Then, we must conclude that $x \in S \cup T$. To do this, we can show that $x \in S \subset S \cup T$ or that $x \in T \subset S \cup T$.

Proof. To show that $S \cup T = R$, we will show the two following set inclusions: $S \cup T \subset R$ and $R \subset S \cup T$.

For the first subset inclusion $S \cup T \subset R$, assume that $x \in S \cup T$. Thus $x \in S$ or $x \in T$. We will proceed with these two cases. In the first case, $x \in S$ and thus $\sin x = 0$. Thus, $\sin x \cos x = 0 \cdot \cos x = 0$ and so $x \in R$. In the second case, $x \in T$ and thus $\cos x = 0$. Thus, $\sin x \cos x = \sin x \cdot 0 = 0$ and so $x \in R$. In either case, we have shown that $x \in R$. Thus, $S \cup T \subset R$.

For the second subset inclusion $R \subset S \cup T$, we will assume that $x \in R$. Thus, x has the property that $\sin x \cos x = 0$. This means that $\sin x = 0$ or $\cos x = 0$, leaving us with two cases. If $\sin x = 0$, then $x \in S \subset S \cup T$ and thus $x \in S \cup T$. If $\cos x = 0$, then $x \in T \subset S \cup T$ and thus $x \in S \cup T$. Either way, $x \in S \cup T$ and thus $R \subset S \cup T$.

Since we have proven above both subset inclusions, we may conclude that $S \cup T = R$.

□

A key technique in the above proof that was used several times is that if we ever assume p or q, then we use two cases: first, arrive at your conclusion using p and then separately, arrive at your conclusion using q.

Below, we will prove some of the more important set theory laws, which state the equalities of certain sets, by using the same technique of proving the two subset inclusions.

2.2.1 DeMorgan's Set Theory Laws

In the previous chapter, we saw that DeMorgan's Logic Laws give us a way to negate conjunctions ("and" statements) and disjunctions ("or" statements). They were:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \text{ and } \neg(p \vee q) \equiv \neg p \wedge \neg q.$$

In Set Theory, there are analogous relationships called **DeMorgan's Set Theory Laws**. These laws are exactly the analogues of the corresponding logic laws, where $\neg$ is replace by the complement, $\wedge$ is replace by $\cap$, and $\vee$ is replaced by $\cup$. DeMorgan's Set Theory Laws are stated below:

$$\overline{S \cap T} = \overline{S} \cup \overline{T} \text{ and } \overline{S \cup T} = \overline{S} \cap \overline{T}.$$

We will prove the latter of these below.

Theorem. Let S and T be sets. Then,

$$\overline{S \cup T} = \overline{S} \cap \overline{T}.$$

Discussion.

`What we want:` $\overline{S \cup T} = \overline{S} \cap \overline{T}$. Thus, we will prove the following two subset inclusions: $\overline{S \cup T} \subset \overline{S} \cap \overline{T}$ and $\overline{S} \cap \overline{T} \subset \overline{S \cup T}$.

`What we'll do:` This proof will be significantly simplified because of DeMorgan's Logic Laws. For this proof, we will use the logic law: $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

Proof. To prove that $\overline{S \cup T} = \overline{S} \cap \overline{T}$, we will prove the following two subset inclusions: $\overline{S \cup T} \subset \overline{S} \cap \overline{T}$ and $\overline{S} \cap \overline{T} \subset \overline{S \cup T}$.

For the first inclusion, assume that $x \in \overline{S \cup T}$. Thus $x \notin S \cup T$. So, it is not true that $x \in S$ or $x \in T$. By DeMorgan's Logic Laws, this is equivalent to $x \notin S$ and $x \notin T$ being true. Thus, since $x \notin S$, then $x \in \overline{S}$, and since $x \notin T$, then $x \in \overline{T}$. Since both $x \in \overline{S}$ and $x \in \overline{T}$, then x is in the intersection: $x \in \overline{S} \cap \overline{T}$. Thus, $\overline{S \cup T} \subset \overline{S} \cap \overline{T}$.

For the second inclusion, assume that $x \in \overline{S} \cap \overline{T}$. Then, $x \in \overline{S}$ and $x \in \overline{T}$. Thus, we have that $x \notin S$ and $x \notin T$. Since both $x \notin S$ and $x \notin T$ is true, we can use DeMorgan's Logic Law to conclude that $x \in S$ or $x \in T$ is not true. Thus, $x \notin S \cup T$, and so $x \in \overline{S \cup T}$. Thus, $\overline{S} \cap \overline{T} \subset \overline{S \cup T}$.

Since we have shown both of these subset inclusions, we know that our two sets are equal: $\overline{S \cup T} = \overline{S} \cap \overline{T}$.

□

2.2.2 The Distributive Laws

The above example of DeMorgan's Set Theory laws clearly indicates that union and intersection, the two major operations of set theory, are related. There remains a question, though, of what occurs if you take some sets and take unions and then intersections. Similar to adding and then multiplying real numbers, a distributive-type property seems to hold. In particular, if S, T, and R are sets, then the following two *distributive laws* hold:

$$S \cup (T \cap R) = (S \cup T) \cap (S \cup R) \text{ and } S \cap (T \cup R) = (S \cap T) \cup (S \cap R).$$

Below, we will prove the first of these two distributive laws.

Theorem. Let S, T, and R be sets. Then,

$$S \cup (T \cap R) = (S \cup T) \cap (S \cup R).$$

Discussion.

`What we want:` $S \cup (T \cap R) = (S \cup T) \cap (S \cup R)$. Thus, we want to show the following two subset inclusions: $S \cup (T \cap R) \subset (S \cup T) \cap (S \cup R)$ and $(S \cup T) \cap (S \cup R) \subset S \cup (T \cap R)$.

`What we'll do:` For the first inclusion $S \cup (T \cap R) \subset (S \cup T) \cap (S \cup R)$, we will assume that $x \in S \cup (T \cap R)$. Thus, because this is an "or"/union assumption, we will have cases: when $x \in S$ and then $x \in T \cap R$. In both cases, we must conclude that $x \in (S \cup T) \cap (S \cup R)$. To show that x is in this intersection, we will need to show both that $x \in S \cup T$ and also that $x \in S \cup R$.

For the second inclusion $(S \cup T) \cap (S \cup R) \subset S \cup (T \cap R)$, we will assume that $x \in (S \cup T) \cap (S \cup R)$. Thus, we are free to assume that both $x \in S \cup T$ and $x \in S \cup R$ are true. To show that $x \in S \cup (T \cap R)$, we will show that $x \in S \subset S \cup (T \cap R)$ or that $x \in (T \cap R) \subset S \cup (T \cap R)$.

Proof. To prove that $S \cup (T \cap R) = (S \cup T) \cap (S \cup R)$, we will prove the two subset inclusions: $S \cup (T \cap R) \subset (S \cup T) \cap (S \cup R)$ and $(S \cup T) \cap (S \cup R) \subset S \cup (T \cap R)$.

For the first inclusion, we will assume that $x \in S \cup (T \cap R)$. Thus, $x \in S$ or $x \in T \cap R$, giving us two cases. If $x \in S$, then $x \in S \subset S \cup T$ and $x \in S \subset S \cup R$. Thus, since x is in both sets, it is in the intersection: $x \in (S \cup T) \cap (S \cup R)$. In the second case, we let $x \in T \cap R$. Thus, $x \in T$ and $x \in R$ are both true. Since $x \in T$, then $x \in T \subset S \cup T$. Since $x \in R$, then $x \in R \subset S \cup R$. Since x is in both sets, it is in the intersection: $x \in (S \cup T) \cap (S \cup R)$. Thus, in either of our two cases, we conclude that $x \in (S \cup T) \cap (S \cup R)$. So, we have obtained the subset inclusion: $S \cup (T \cap R) \subset (S \cup T) \cap (S \cup R)$.

For the second inclusion, we will assume that $x \in (S \cup T) \cap (S \cup R)$. Thus, we know that $x \in S \cup T$ and also $x \in S \cup R$. Since $x \in S \cup T$, then $x \in S$ or $x \in T$; furthermore, since $x \in S \cup R$, then $x \in S$ or $x \in R$. So, at least one of $x \in S$ or $x \in T$ has to be true while, at the same time, at least one of $x \in S$ or $x \in R$ must be true. If we know that $x \in S$, then $x \in S \subset S \cup (T \cap R)$. If it's not true that $x \in S$, then it must be that $x \in T$ and that $x \in R$. Thus, $x \in T \cap R$ and thus $x \in (T \cap R) \subset S \cup (T \cap R)$. So, we have the subset inclusions $(S \cup T) \cap (S \cup R) \subset S \cup (T \cap R)$.

Since we have shown both inclusions to be true, then we can conclude the two sets are equal: $S \cup (T \cap R) = (S \cup T) \cap (S \cup R)$.

□

2.3 Product Sets

Since we are interested in building new sets from old sets, we can look at the example of two-dimensional real space $\mathbb{R}^2$. This set is geometrically represented by the xy-plane, and individual elements in the plane take on the form of the 2-tuple (x, y) for $x, y \in \mathbb{R}$. Thus, we can write $\mathbb{R}^2$ in set-builder notation as

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}.$$

Thus, $\mathbb{R}^2$ can be thought of as two copies of $\mathbb{R}$, and elements in this set have two entries: one for each real number.

2.3.1 Examples of Product Sets

If we wish to generalize the example of $\mathbb{R}^2$ to that for two different sets S and T, then we can form what is called the **direct product** or **cartesian product** of the sets S and T. This new, larger set is denoted by $S \times T$ and is given by

$$S \times T = \{(s, t) \mid s \in S \text{ and } t \in T\}.$$

It is important to note that individual elements in the product $S \times T$ look like the 2-tuple (s, t) and that we know that the first entry is an element of S and the second entry is an element of T: $s \in S$ and $t \in T$. Below are some examples of products.

- If we take $S = T = \mathbb{R}$, then we have that

  $$\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\},$$

  which is exactly the same as $\mathbb{R}^2$.

- If $S = \{a, b\}$ is a 2-element set and $T = \{x, y, z\}$ is a 3-element set, then the product space $S \times T$ is set of all possible tuples (s, t) with $s \in S$ and $t \in T$. Thus, $S \times T$ contains a total of 6 elements:

  $$S \times T = \{(a, x), (a, y), (a, z), (b, x), (b, y), (b, z)\}.$$

- In the same sprit as $\mathbb{R} \times \mathbb{R}$, the product of two copies of $\mathbb{Z}$ is called an integer lattice and is given by

  $$\mathbb{Z} \times \mathbb{Z} = \{(n, m) \mid n, m \in \mathbb{Z}\}.$$

  Geometrically, this corresponds to the points in $\mathbb{R}^2$ that have both of their coordinates integers and appears as a lattice of points in the plane.

- Notice that, in general, the products $S \times T$ and $T \times S$ are different sets. For example, if we use the S and T from the second example above, we have that

$$T \times S = \{(x,a),(x,b),(y,a),(y,b),(z,a),(z,b)\}.$$

The sets $S \times T$ and $T \times S$ are not equal because, for example, the element $(a,x) \in S \times T$ is completely different from the $(x,a) \in T \times S$. In fact, $(a,x) \notin T \times S$ since it is not true that $a \in T$ (nor is it true that $x \in S$).

- If S is any set and $\varnothing$ is the empty set, then notice that $\varnothing \times S = \varnothing$ and $S \times \varnothing = \varnothing$. This follows from the fact that, for example, $S \times \varnothing$ is the set of all element (s,t) such that $s \in S$ and $t \in \varnothing$. Since there are no elements in $\varnothing$, there are no tuples in the product $S \times \varnothing$.

- We can also form triple (or any number of) products. If we take three sets S, T, and R, then

$$S \times T \times R = \{(s,t,r) \mid s \in S, t \in T, \text{ and } r \in R\}.$$

For triple products, a general element looks like a 3-tuple (s,t,r).

2.3.2 Proofs involving Product Sets

Now that we have this new structure called direct product, we can ask how this interacts with the various other set theoretic operations. Below, we will prove that intersection and direct products behave nicely with respect to each other. In fact, if A, B, C and D are sets, then we will show that

$$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D).$$

Proposition. Let A, B, C, and D be sets. Then

$$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D).$$

Discussion.

`What we want:` $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$. Thus, we will need to show the two subset inclusions:

$$(A \cap B) \times (C \cap D) \subset (A \times C) \cap (B \times D) \text{ and}$$

$$(A \times C) \cap (B \times D) \subset (A \cap B) \times (C \cap D).$$

`What we'll do:` When we use direct products in proofs, it's important to keep in mind that our arbitrary elements will look like 2-tuples (x,y). Thus, for example, when we do our first subset inclusion and take an arbitrary element in $(A \cap B) \times (C \cap D)$, it will look like (x,y) such that $x \in A \cap B$ and $y \in C \cap D$.

Proof. To show that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$, we will show the following two subset inclusions: $(A \cap B) \times (C \cap D) \subset (A \times C) \cap (B \times D)$ and $(A \times C) \cap (B \times D) \subset (A \cap B) \times (C \cap D)$.

For the first inclusion, assume that $(x,y) \in (A \cap B) \times (C \cap D)$. Thus, $x \in A \cap B$ and $y \in C \cap D$. Since $x \in A \cap B$, then $x \in A$ and $x \in B$. Since $y \in C \cap D$, then $y \in C$ and $y \in D$. So, since $x \in A$ and $y \in C$, we know that $(x,y) \in A \times C$. Furthermore, since $x \in B$ and $y \in D$, then we also know that $(x,y) \in B \times D$. Since (x,y) is in both sets, we can conclude that $(x,y) \in (A \times C) \cap (B \times D)$, and thus $(A \cap B) \times (C \cap D) \subset (A \times C) \cap (B \times D)$.

For the second inclusion, assume that $(x, y) \in (A \times C) \cap (B \times D)$. Thus, $(x, y) \in A \times C$ and $(x, y) \in B \times D$. Since $(x, y) \in A \times C$, we know that $x \in A$ and $y \in C$. Furthermore, since $(x, y) \in B \times D$, we know that $x \in B$ and $y \in D$. Since $x \in A$ and $x \in B$, then $x \in A \cap B$. Also, since $y \in C$ and $y \in D$, then $y \in C \cap D$. So, $(x, y) \in (A \cap B) \times (C \cap D)$. Thus, $(A \times C) \cap (B \times D) \subset (A \cap B) \times (C \cap D)$.

Since we have shown both subset inclusions, we can conclude that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

$\square$