Individual Round 2024-2025 Solutions

Problem 1. Caltech Hall has 9 floors. An elevator starts at floor 1 and goes up to floor 9, stopping at every floor. The numbers 2 through 9 are randomly distributed among four people so that each person gets two of them. Each person gets on the elevator at their lower number and gets off at their higher number. What is the probability that everyone is in the elevator at once?

Proposed by Vivian Loh

Solution: $\boxed{\frac{8}{35}}$.

We don't really care who gets which pair of floor numbers; we just care about the partitioning of the 8 floor numbers into 4 pairs. The total number of ways to pair up the eight numbers 2-9 into 4 pairs is

$$\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{4!} = 7 \cdot 5 \cdot 3 \cdot 1 = 105.$$

In order for everyone to be in the elevator at once, 2, 3, 4, and 5 must be the 4 lower numbers over all the pairs and 6, 7, 8, 9 must be the higher numbers over all the pairs. Thus, there are $4! = 24$ different valid partitionings of the eight numbers into pairs, since each of $\{6,7,8,9\}$ must be assigned to one of $\{2,3,4,5\}$. So the answer is $\frac{24}{105} = \boxed{\frac{8}{35}}$.

Problem 2. Find the smallest positive integer n such that $n \cdot 2^{17} + 1$ has a divisor d such that $d+1$ is divisible by 2^{12}.

Proposed by Justin Lee

Solution: $\boxed{3967}$.

We have that $a \cdot 2^{12} - 1 \mid n \cdot 2^{17} + 1$. The quotient must be $-1 \mod 2^{12}$, so we can write $n \cdot 2^{17} + 1 = (a \cdot 2^{12} - 1)(b \cdot 2^{12} - 1)$. This implies that $2^5 \mid a + b$, so the smallest possible value of n is such that

$$n \cdot 2^{17} + 1 = (2^{12} - 1)(31 \cdot 2^{12} - 1) \implies n = 3967.$$

Problem 3. Find the largest positive integer $n \neq 2025$ such that $nx - \lfloor \sqrt{n} \lfloor \sqrt{n} \lfloor x \rfloor \rfloor \rfloor \leq 2025$ holds for all real x. (Here $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.)

Proposed by Justin Lee

Solution: $\boxed{1980}$.

Observe that $x - \lfloor x \rfloor < 1 \implies \sqrt{n}x - \sqrt{n}\lfloor x \rfloor < \sqrt{n} \implies \sqrt{n}x - \lfloor \sqrt{n} \lfloor x \rfloor \rfloor < \sqrt{n} + 1 \implies$ $nx - \sqrt{n}\lfloor \sqrt{n} \lfloor x \rfloor \rfloor < n + \sqrt{n} \implies nx - \lfloor \sqrt{n} \lfloor \sqrt{n} \lfloor x \rfloor \rfloor \rfloor < n + \sqrt{n} + 1$. However, the function $f(x) = nx - \lfloor \sqrt{n} \lfloor \sqrt{n} \lfloor x \rfloor \rfloor \rfloor$ is monotonic in each interval $[c, c+1)$ where c is an integer. Thus, the supremum value of $f(x)$ must be of the form $n(c+1) - \lfloor \sqrt{n} \lfloor c\sqrt{n} \rfloor \rfloor$, which is in particular an integer. It hence follows that $f(x) \leqslant \lfloor n + \sqrt{n} + 1 \rfloor$. It remains to show that this is the least integral upper bound for $f(x)$ whenever n is not a perfect square. Indeed, for any $\varepsilon > 0$ there exists an integer c such that $c\sqrt{n} - \lfloor c\sqrt{n} \rfloor > 1 - \varepsilon$. Thus for the value $z = c + 1 - \varepsilon$ we have $f(z) \geqslant nz - \sqrt{n}\lfloor c\sqrt{n} \rfloor > n(1-\varepsilon) + \sqrt{n}(1-\varepsilon)$. Thus, any upper bound for $f(x)$ must be at least $n + \sqrt{n}$, as desired.

To conclude, note that $\lfloor n + \sqrt{n} + 1 \rfloor \leq 2025$ is satisfied exactly for $n \leq 1980$. Thus, $\boxed{1980}$ is our answer.

Problem 4. In rectangle $ABCD$, points E and F are chosen on sides BC and CD respectively such that EF bisects $\angle CFA$ and $\angle AEF = 90^\circ$. Given that $CF = 9$ and $FD = 7$, what is the area of triangle AEF?

Proposed by Vivian Loh and Justin Lee

Solution: $\boxed{150}$.
$\angle CFE = \angle AFE \implies \triangle AEF \sim \triangle ECF$, so the circumcircle of $\triangle AEF$ is tangent to BC. Thus, the center of this circle (which is also the midpoint of AF since AF is a diameter) lies on the line through E perpendicular to BC. So, $BE = EC = x$. Also, $\angle FEC + \angle BEA = 90^\circ \implies \triangle CEF \sim \triangle BAE$, so $AB = \frac{x^2}{9}$. Since $CD = AB = 16$, we have $\frac{x^2}{9} = 16 \implies x = 12$, and by the Pythagorean Theorem we get $EF = 15, AE = 20$, so the area of $\triangle AEF$ is $\frac{15 \cdot 20}{2} = \boxed{150}$.

Problem 5. Find the number of odd positive integers $n < 1000$ such that if S_n denotes the set of divisors of n, then exactly half of the elements of S_n are divisible by 3, exactly a third of the elements of S_n leave a remainder of 1 upon dividing by 3, and exactly a sixth of the elements of S_n leave a remainder of 2 upon dividing by 3.

Proposed by Justin Lee

Solution: $\boxed{5}$.
The condition that half of the elements of S_n are divisible by 3 is equivalent to $3 \mid n$ and $9 \nmid n$. Now, write $n/3 = m \cdot \prod_{i=1}^{k} p_i^{a_i}$, where all divisors of m are $1 \pmod 3$ and $p_i \equiv 2 \pmod 3$. Then,

$$\prod_{i=1}^{k} \sum_{j=0}^{a_i} (-1)^j$$

counts the number of $1 \pmod 3$ elements of S_n subtracted by the number of $2 \pmod 3$ elements of S_n. Hence, we have

$$\prod_{i=1}^{k} \frac{\sum_{j=0}^{a_i} (-1)^j}{\sum_{j=0}^{a_i} 1^j} = \frac{1}{3}$$

which is only possible when $k = 1$ and $a_1 = 2$. Thus, n satisfies the desired conditions if and only if n is of the form $3p^2m$ where $p \equiv 2 \pmod 3$ and all prime divisors of m are $1 \pmod 4$. The possible values of p are $5, 11, 17$, yielding 5 total possible numbers: $75 \cdot 1$, $75 \cdot 7$, $75 \cdot 13$, $3 \cdot 11^2$, and $3 \cdot 17^2$.

Problem 6. Determine the value of

$$\prod_{n=1}^{\infty} \left(\frac{1 + 2\cos\left(\frac{\pi}{3^n}\right)}{3} \right) = \frac{1 + 2\cos\left(\frac{\pi}{3}\right)}{3} \cdot \frac{1 + 2\cos\left(\frac{\pi}{9}\right)}{3} \cdot \frac{1 + 2\cos\left(\frac{\pi}{27}\right)}{3} \cdots .$$

Proposed by Shreyas Singh

Solution: $\boxed{\frac{2}{\pi}}$.
Since $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$, the product is equal to

$$\prod_{n=1}^{\infty} \left(\frac{1 + e^{i \cdot \frac{\pi}{3^n}} + e^{-i \cdot \frac{\pi}{3^n}}}{3} \right) = \left(\frac{1 + e^{\frac{i\pi}{3}} + e^{-\frac{i\pi}{3}}}{3} \right) \left(\frac{1 + e^{\frac{i\pi}{9}} + e^{-\frac{i\pi}{9}}}{3} \right) \cdots .$$

Let $\alpha = e^{i\pi}$. Then this is equal to

$$\left(\frac{1 + \alpha^{\frac{1}{3}} + \alpha^{-\frac{1}{3}}}{3} \right) \left(\frac{1 + \alpha^{\frac{1}{9}} + \alpha^{-\frac{1}{9}}}{3} \right) \left(\frac{1 + \alpha^{\frac{1}{27}} + \alpha^{-\frac{1}{27}}}{3} \right) \cdots$$

$$= \left(\frac{\alpha^0+\alpha^{\frac{1}{3}}+\alpha^{-\frac{1}{3}}}{3}\right)\left(\frac{\alpha^0+\alpha^{\frac{1}{9}}+\alpha^{-\frac{1}{9}}}{3}\right)\left(\frac{\alpha^0+\alpha^{\frac{1}{27}}+\alpha^{-\frac{1}{27}}}{3}\right)\cdots$$

$$= \alpha^{-0.\overline{1}_3}\cdot\alpha^{0.\overline{1}_3}\left(\frac{\alpha^0+\alpha^{0.1_3}+\alpha^{-0.1_3}}{3}\right)\left(\frac{\alpha^0+\alpha^{0.01_3}+\alpha^{-0.01_3}}{3}\right)\left(\frac{\alpha^0+\alpha^{0.001_3}+\alpha^{-0.001_3}}{3}\right)\cdots$$

which is $\alpha^{-0.\overline{1}_3}$ times the average of $\alpha^{0.abcd\ldots_3}$ for all infinitely long decimals $0.abcd\ldots$ between 0 and 1 in base 3. Thus, it is equal to

$$\alpha^{-0.\overline{1}_3}\int_0^1 \alpha^x\,dx = \alpha^{-0.\overline{1}_3}\frac{\alpha^x}{\ln(\alpha)}\bigg|_0^1 = \alpha^{-0.\overline{1}_3}\left(\frac{\alpha-1}{\ln(\alpha)}\right).$$

$0.\overline{1}_3 = \frac{1}{2}$, so this is

$$\frac{\alpha^{1/2}-\alpha^{-1/2}}{\ln(\alpha)} = \frac{e^{i\frac{\pi}{2}}-e^{-i\frac{\pi}{2}}}{i\pi} = \frac{2i\sin\frac{\pi}{2}}{i\pi} = \frac{2\sin\frac{\pi}{2}}{\pi} = \boxed{\frac{2}{\pi}}.$$

Problem 7. On an infinite lattice grid, a circle with radius $0 < r < 10$ is drawn, centered at the origin. Each grid square is shaded if and only if the boundary of the circle passes through its interior (not only a corner or side). Let N be the number of grid squares that are shaded. How many possible values of N are there?

Proposed by Justin Lee and Vivian Loh

Solution: $\boxed{17}$.

First consider circles which don't pass through any lattice points that aren't on the $x-$ or $y-$ axes. The number of grid squares in the first quadrant that it passes through is 1 plus the number of grid lines that it crosses. If $n < r \le n+1$, then there are $2n$ such grid lines (n vertical and n horizontal), so the circle passes through a total of $2n+1$ grid squares in the first quadrant. Since we multiply by 4 for the four quadrants, we get a total of $4(2n+1)$ grid squares. Thus, we can achieve $N = 4\cdot 1, 4\cdot 3, 4\cdot 5, \ldots, 4\cdot 19$.

Now consider the case in which the circle passes through a lattice point which is not on the $x-$ or $y-$ axes. This means that the circle intersects two grid lines in the first quadrant simultaneously, (one vertical and horizontal), so we must subtract one from the formula obtained from the previous case, 1+(number of grid lines crossed), for each such lattice point. Note that if the number of lattice points crossed in the first quadrant is even, then $2n+1$ minus an even amount will just be $2m+1$ for some $m < n$, so we won't get any new values of N outside of the set $\{4\cdot 1, 4\cdot 3, \ldots, 4\cdot 19\}$. Thus, we only have to consider the case in which the circle passes through an odd number of lattice points in the first quadrant, which by symmetry about the line $x = y$ is equivalent to $r = x\sqrt{2}$ for some integer x. Since $r < 10$, the possible values of r are $\sqrt{2}, 2\sqrt{2}, \ldots, 7\sqrt{2}$, which each yield a different, even, value of $\frac{N}{4}$; $3-1=2$, $5-1=4$, $9-1=8$, $11-1=10$, $15-1=14$, $17-1=16$, $19-1=18$ respectively.

Thus, the total set of possible values of N is

$$\{4\cdot 1, 4\cdot 2, 4\cdot 3, 4\cdot 4, 4\cdot 5, 4\cdot 7, 4\cdot 8, 4\cdot 9, 4\cdot 10, 4\cdot 11, 4\cdot 13, 4\cdot 14, 4\cdot 15, 4\cdot 16, 4\cdot 17, 4\cdot 18, 4\cdot 19\},$$

a total of $\boxed{17}$ values.

Problem 8. Points A, B, and C lie on a line in that order, and point D is such that $\angle DBC = 60^\circ$ and $\angle ADB + \angle ADC = 150^\circ$. Point E is on the circumcircle of $\triangle BCD$ such that line BE is tangent to the circumcircle of $\triangle ABD$. If $DE = 4$ and $CE = 3$, what is BE?

Proposed by Vivian Loh

Solution: $\boxed{\frac{10\sqrt{3}}{3}}$

Angle chasing gives that $\triangle DBA \sim \triangle DEC$. (Note that this also implies $\triangle DAC \sim \triangle DBE$.) Let E' be the foot of the altitude from E to line AD; then $EE' = 2$, and C, E, E' are collinear because $\angle DEC = 120^\circ$ and $\angle DEE' = 60^\circ$. $\triangle E'DC \sim \triangle E'CA$, so $AC = CD \cdot \frac{CE'}{DE'}$. Note that since $\triangle DAC \sim \triangle DBE$, $BE = AC \cdot \frac{DE}{CD} = DE \cdot \frac{CE'}{DE'} = 4 \cdot \frac{5}{2\sqrt{3}} = \frac{10}{\sqrt{3}} = \boxed{\frac{10\sqrt{3}}{3}}$.

Problem 9. Let $\circ$ be a binary operation on the real numbers satisfying the properties

$$(ab) \circ (ac) = |a|(b \circ c) \qquad \text{for all real numbers } a, b, c$$

and

$$(a \circ b) \circ (c \circ d) = (a+b) \circ (c+d) \qquad \text{for all real numbers } a, b, c, d \text{ satisfying } ab + cd = 0$$

Find the product of all possible nonzero values of $20 \circ 25$.

Proposed by Justin Lee

Solution: -1025.

Note that $-c \circ -d = c \circ d$. Thus, whenever $ab + cd = 0$, we have that $(a+b) \circ (-c-d) = (a \circ b) \circ (-c \circ -d) = (a \circ b) \circ (c \circ d) = (a+b) \circ (c+d)$. It follows that $m \circ n = m \circ -n$ for any real m, n. Consequently, for $ab + cd = 0$,

$$(a+b) \circ (c+d) = (a \circ b) \circ (c \circ d) = (a \circ -b) \circ (c \circ -d) = (a-b) \circ (c-d)$$

Now for any real numbers x, y, z such that $|x| \geqslant |z|$, we can find real numbers a, b, c, d such that $a + b = x$ and $c + d = y$ and $a - b = z$ and $ab + cd = 0$. Indeed, the numbers $a = \frac{x+z}{2}$ and $b = \frac{x-z}{2}$ satisfy $ab \geqslant 0$, so there exists a solution (c, d) to $c + d = y$ and $cd = -ab$.

Since $|c - d| = \sqrt{y^2 + 4ab} = \sqrt{x^2 + y^2 - z^2}$, we deduce that $x \circ y = z \circ \sqrt{x^2 + y^2 - z^2}$ whenever $|x| \geqslant |z|$. In particular, for any (z, w) such that $|z| \leqslant |w|$, we have that

$$\sqrt{\frac{z^2 + w^2}{2}} \circ \sqrt{\frac{z^2 + w^2}{2}} = z \circ w$$

Finally, note that for any real numbers m, n,

$$(m \circ n) \circ (m \circ n) = (m \circ n) \circ (m \circ -n) = (m+n) \circ (m-n)$$

If we let $1 \circ 1 = k$, then $k|m \circ n| = (m+n) \circ (m-n)$ and thus $k|(m+n) \circ (m-n)| = 2m \circ 2n$. This implies that $k = 0$ or $k = \pm\sqrt{2}$, and thus (the nonzero values are) $20 \circ 25 = \pm\sqrt{20^2 + 25^2}$. Both values are achievable, on account of the functions $x \circ y = \sqrt{x^2 + y^2}$ and $x \circ y = -\sqrt{x^2 + y^2}$. The answer is $-(20^2 + 25^2) = -1025$.

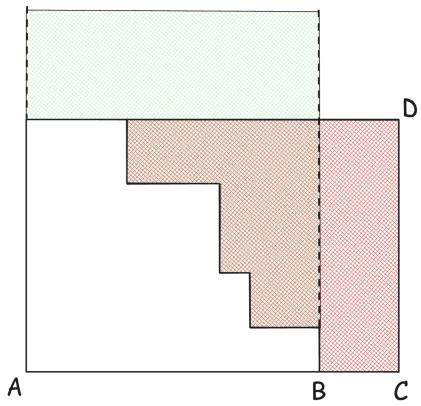
Problem 10. Hannah has a 2024×2025 rectangle in the coordinate plane, with sides parallel to the axes. She makes a cut from one side to another side which only goes down and/or right along grid lines. Then she puts the two pieces together, possibly with rotations and/or reflections without overlaps or gaps, to form a new rectangle which is not congruent to the original. How many possible new rectangles can she produce? (An $a \times b$ rectangle is considered the same as a $b \times a$ rectangle.)

Proposed by Vivian Loh

Solution: $\boxed{7}$.

Claim: The cut must either bisect the rectangle, or be a zigzag from one corner to the opposite corner with each step the same width and height.

Clearly, the cut must go between opposite edges of the rectangle, otherwise three of the rectangle's sides are preserved and so it must be the original rectangle. So, either it is one straight cut or it is a zigzag like this one (A, C, and D are vertices of the original rectangle) that splits the rectangle into the white region and the pink region.



Suppose the white region is fixed in that orientation (we can rotate/reflect the rectangle we end up with so that this is the case). Then in order for the pink region to form a rectangle with the white region that does not have a side length equal to 2024 or 2025, AB must be a side of the new rectangle. This means that the pink region must go in the green region, between the two dotted lines. Now we can either translate the pink region to fit in that area, or reflect it about the bisector of angle $\angle D$ and then translate it. The former results in all of the steps being the same width and height, and the latter would mean $CD = AB$, which is a contradiction because it implies the resultant rectangle would be congruent to the original one. So the claim is proven.

In both the bisection case and the zigzag case, one dimension is multiplied by $\frac{p}{q}$ and the other by $\frac{q}{p}$, where p and q are consecutive positive integers (the bisection case is p, q = 1, 2). Furthermore, if one of the dimensions is divisible by q and the other by p, then there will always be a zigzag that achieves new dimensions that are the old ones multiplied by $\frac{p}{q}$ and $\frac{q}{p}$. Thus, the new dimensions of the rectangle must $\left(2025 \cdot \frac{p}{q}\right) \times \left(2024 \cdot \frac{q}{p}\right)$ where p and q are consecutive positive integers. Since $q \mid 2025$ and $p \mid 2024$, by factorization we find that the only possible pairs (p,q) are $(2,1)$, $(2,3)$, $(4,3)$, $(4,5)$, $(8,9)$, $(44,45)$, $(46,45)$, and $(2024,2025)$. However, the last one would result in the new rectangle being congruent to the original (2024×2025), and $(44,45)$ actually results in the same new triangle as $(45,46)$, since $2024 \cdot \frac{45}{44} = 45 \cdot 46 = 2025 \cdot \frac{46}{45}$. Each of the other 6 pairs determines a unique new rectangle which is not congruent to the original, so the answer is $\boxed{6}$.

We had the answer wrong during the actual competition, and accepted the answer of 7.

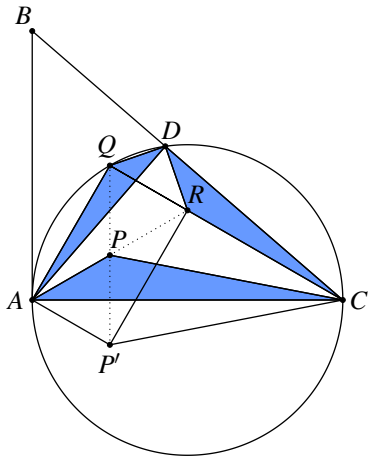
Problem 11. In triangle $\triangle ABC$, $\angle BAC = 90^\circ$. D is the foot of the altitude from A to BC. Point P is inside $\triangle ABC$ such that $\angle PAC = \angle PCB$. Points Q and R are inside $\triangle ABD$ and $\triangle ACD$, respectively, such that $\triangle PAC \sim \triangle QDA \sim \triangle RDC$. If $\triangle PQR$ is equilateral, what is $\frac{AB}{AC}$?

Proposed by Vivian Loh

Solution: $\boxed{\frac{\sqrt{3}}{2}}$.

Note that $PABC \sim QDBA \sim RDAC$. Thus, $\angle QAB = \angle QDA \implies (QDA)$ is tangent to AB, which means that

$QDCA$ is cyclic. Thus, $\angle DCQ = \angle DAQ = \angle DCR \implies$ Q, R, and C are collinear.



Let P' be the reflection of P over AC. Then $\triangle CRD \sim \triangle CP'A$, so by Spiral Similarity, $\triangle CDA \sim \triangle CRP' \implies \angle QRP' = 90^\circ$. We also know that $\angle RQA = 90^\circ$, and $\angle QAP' = 90^\circ$ since $\angle BAQ = \angle P'AC$, so $QRP'A$ is a rectangle. If $\triangle PQR$ is equilateral, then $PQ = PR \implies P$ lies on the perpendicular bisector of QR, so $PA = PP' \implies \triangle PAP'$ is also equilateral. Thus, the sides of the rectangle are in the ratio $QR : AQ = 1 : \sqrt{3}$, and $\angle PAC = 30^\circ$. Thus, $\angle QDA = \angle QCA = 30^\circ$, so $AQ : QC = \sqrt{3} : 3$, so $P'R : RC = \sqrt{3} : 2$. $\triangle CRP' \sim \triangle CDA \sim \triangle CAB$, so $\frac{AB}{AC} = \frac{P'R}{RC} = \boxed{\frac{\sqrt{3}}{2}}$.

Problem 12. Suppose x, y, and z are complex numbers satisfying

$$|xyz| = 12$$
$$|x\bar{y} + z| = \frac{15\sqrt{3}}{2}$$
$$|y\bar{z} + x| = 3\sqrt{6}$$

Compute the minimum possible value of $|z\bar{x} + y|$.

(Here, $\bar{w}$ denotes the complex conjugate of w and $|w|$ denotes the norm of w, where $|a+bi| = \sqrt{a^2+b^2}$.)

Proposed by Justin Lee

Solution: $\boxed{\sqrt{2}}$.

For convenience let $|x|^2 = a$, $|y|^2 = b$, and $|z|^2 = c$, so that $abc = 144$. Also, let $xyz = p$. The second equation tells us

$$|x\bar{y} + z|^2 = (x\bar{y} + z)(\bar{x}y + z) = |xy|^2 + |z|^2 + \bar{x}yz + x\bar{y}z = \frac{144}{c} + c + \frac{2\mathrm{Re}(\bar{p}x^2)}{a^2} = \frac{675}{4}. \tag{1}$$

Similarly, the third equation expands as

$$\frac{144}{a} + a + \frac{2\mathrm{Re}(\bar{p}y^2)}{b^2} = 54 \tag{2}$$

Since $\mathrm{Re}(w)$ is bounded between $-|w|$ and $|w|$ for any number w, it follows that $-24 \leqslant \frac{2\mathrm{Re}(\bar{p}x^2)}{a^2} \leqslant 24$ and $-24 \leqslant \frac{2\mathrm{Re}(\bar{p}y^2)}{b^2} \leqslant 24$ so the second and third equations imply the conditions

$$\frac{579}{4} \leqslant \frac{144}{c} + c \leqslant \frac{771}{4} \qquad \text{and} \qquad 30 \leqslant \frac{144}{a} + a \leqslant 78 \tag{3}$$

We wish to minimize $|y\bar{z}+x|^2$. We have that

$$|y\bar{z}+x|^2 = \frac{144}{b} + b + \frac{2\mathrm{Re}(\bar{p}z^2)}{c^2} \geqslant \frac{144}{b} + b - 24 \tag{4}$$

Now I claim that for any (a,b,c) satisfying $abc = 144$ and 3, we can find x,y,z satisfying $|x|^2 = a$, $|y|^2 = b$, $|z|^2 = c$ and the original equations so that equality is achieved in 4. Indeed, by 1 and 2, the original equations are satisfied by setting $\bar{p}x^2$ and $\bar{p}y^2$ to some particular values; under these constraints, there exists a tuple (x,y,z) such that $\bar{p}z^2 = -12c^2$, which implies equality in 4.

Consequently, the problem reduces to minimizing $\frac{144}{b} + b$ while subject to the constraints 3 as well as $abc = 144$. To do that, we wish to make b as close to 12 as possible, so we want to choose a and c so that ac is as close to 12 as possible. The inequalities 3 imply $a \in [\approx 1.89, 6] \cup [24, \approx 76.1]$ and $c \in [\frac{3}{4}, \approx 1.002] \cup [\approx 143.75, 192]$. It is clear that optimally we should pick $a = 24$ and $c = \frac{3}{4}$, which gives $b = 8$ and by 4, $|y\bar{z}+x|^2 \geqslant \frac{144}{8} + 8 - 24 = 2$. Hence, the answer is $\boxed{\sqrt{2}}$.