CMM 2025 Integration Bee Final Round Hard Solutions

Problem 1. $\int -e^x\sqrt{4e^{2x}-9}\ \mathrm{dx}$

Solution: $\boxed{\frac{9}{4}\ln\left|\frac{2}{3}e^x+\frac{1}{3}\sqrt{4e^{2x}-9}\right|-\frac{1}{2}e^x\sqrt{4e^{2x}-9}+C}$

Immediately from the presence of an e^x dx term and from the form of the square root argument, we notice that some trigonometric substitution mapping to e^x will be necessary. We consider $e^x = \frac{3}{2}\sec(\theta)$ because of a Pythagorean identity simplification to come. Thus, $e^x\ \mathrm{dx} = \frac{3}{2}\sec(\theta)\tan(\theta)\ \mathrm{d}\theta$. Making our substitution, we gain

$$\int -e^x\sqrt{4e^{2x}-9}\ \mathrm{dx} = \int -\sqrt{4e^{2x}-9}(e^x\ \mathrm{dx}) = \int -\sqrt{4\left(\frac{9}{4}\sec^2(\theta)\right)-9}\left(\frac{3}{2}\sec(\theta)\tan(\theta)\ \mathrm{d}\theta\right).$$

After simplifying, we exploit the trigonometric identity $\tan^2(\theta) = \sec^2(\theta)-1$, as shown.

$$\frac{3}{2}\int -\sec(\theta)\tan(\theta)\sqrt{9\sec^2(\theta)-9}\ \mathrm{d}\theta = \frac{9}{2}\int -\sec(\theta)\tan(\theta)\sqrt{\sec^2(\theta)-1}\ \mathrm{d}\theta = -\frac{9}{2}\int \sec(\theta)\tan^2(\theta)\ \mathrm{d}\theta.$$

We justify that $\tan(\theta) = \sqrt{\sec^2(\theta)-1}$ because our initial substitution $e^x = \frac{3}{2}\sec(\theta)$ ensures $|\tan(\theta)| = \tan(\theta)$ for all θ. Once again using $\tan^2(\theta) = \sec^2(\theta)-1$,

$$-\frac{9}{2}\int \sec(\theta)\tan^2(\theta)\ \mathrm{d}\theta = -\frac{9}{2}\int \sec(\theta)(\sec^2(\theta)-1)\ \mathrm{d}\theta = -\frac{9}{2}\int \sec^3(\theta)\ \mathrm{d}\theta + \frac{9}{2}\int \sec(\theta)\ \mathrm{d}\theta.$$

At this point, it will be helpful to designate our goal integral as

$$I = -\frac{9}{2}\int \sec(\theta)\tan^2(\theta)\ \mathrm{d}\theta.$$

Then,

$$I = -\frac{9}{2}\int \sec^3(\theta)\ \mathrm{d}\theta + \frac{9}{2}\int \sec(\theta)\ \mathrm{d}\theta = -\frac{9}{2}\int \sec^2(\theta)\sec(\theta)\ \mathrm{d}\theta + \frac{9}{2}\int \sec(\theta)\ \mathrm{d}\theta.$$

And with $\sec^3(\theta)$ modified into $\sec^2(\theta)\sec(\theta)$, we perform integration by parts on such, giving

$$I = -\frac{9}{2}\left(\sec(\theta)\tan(\theta) - \int \sec(\theta)\tan^2(\theta)\ \mathrm{d}\theta\right) + \frac{9}{2}\int \sec(\theta)\ \mathrm{d}\theta.$$

We notice

$$\frac{2}{9}I = -\int \sec(\theta)\tan^2(\theta)\ \mathrm{d}\theta,$$

so, substituting this, we have

$$I = -\frac{9}{2}\left(\sec(\theta)\tan(\theta) + \frac{2}{9}I\right) + \frac{9}{2}\int \sec(\theta)\ \mathrm{d}\theta$$

$$I = -\frac{9}{2}\sec(\theta)\tan(\theta) - I + \frac{9}{2}\int \sec(\theta)\ \mathrm{d}\theta$$

$$I = -\frac{9}{4}\sec(\theta)\tan(\theta) + \frac{9}{4}\int \sec(\theta)\ \mathrm{d}\theta.$$

The integral of secant is well-known; thus, we find that

$$I = \frac{9}{4}\ln|\sec(\theta)+\tan(\theta)| - \frac{9}{4}\sec(\theta)\tan(\theta) + C.$$

To gain an expression in terms of x, we recall $e^x = \frac{3}{2}\sec(\theta)$, meaning $\sec(\theta) = \frac{2}{3}e^x$. Moreover, $\tan(\theta) = \tan\left(\operatorname{arcsec}\left(\frac{2}{3}e^x\right)\right) = \frac{1}{3}\sqrt{4e^{2x}-9}$, which can be determined geometrically through letting $\theta = \operatorname{arcsec}\left(\frac{2}{3}e^x\right)$ be an angle in a right triangle, thereafter analyzing the trigonometric correlation among sides. With this, we revert our initial substitution;

$$I = \frac{9}{4}\ln|\sec(\theta)+\tan(\theta)| - \frac{9}{4}\sec(\theta)\tan(\theta) + C = \frac{9}{4}\ln\left|\frac{2}{3}e^x + \frac{1}{3}\sqrt{4e^{2x}-9}\right| - \frac{9}{4}\left(\frac{2}{3}e^x\right)\left(\frac{1}{3}\sqrt{4e^{2x}-9}\right) + C.$$

And finally,

$$I = \boxed{\frac{9}{4}\ln\left|\frac{2}{3}e^x + \frac{1}{3}\sqrt{4e^{2x}-9}\right| - \frac{1}{2}e^x\sqrt{4e^{2x}-9} + C}.$$

Problem 2. $\displaystyle\int \frac{\sqrt[3]{x}}{1+x}\ \mathrm{dx}$

Solution: $\boxed{3\sqrt[3]{x} - \ln|\sqrt[3]{x}+1| + \frac{1}{2}\ln\left|x^{\frac{2}{3}} - \sqrt[3]{x}+1\right| - \sqrt{3}\arctan\left(\frac{2}{\sqrt{3}}\sqrt[3]{x} - \frac{1}{\sqrt{3}}\right) + C}$

It will prove useful for this evaluation that we let $u^3 = x$. Accordingly, $3u^2\ \mathrm{du} = \mathrm{dx}$. Substituting using these items yields

$$\int \frac{\sqrt[3]{x}}{1+x}\ \mathrm{dx} = \int \frac{u3u^2}{u^3+1}\ \mathrm{du} = 3\int \frac{u^3}{u^3+1}\ \mathrm{du}.$$

We now add 0 to our integrand in a special form, as follows.

$$3\int \frac{u^3}{u^3+1}\ \mathrm{du} = 3\int \frac{u^3+1-1}{u^3+1}\ \mathrm{du} = 3\int \mathrm{du} - 3\int \frac{1}{u^3+1}\ \mathrm{du} = 3u - 3\int \frac{1}{u^3+1}\ \mathrm{du}.$$

We notice that our denominator of our current integrand is simply a sum of cubes, meaning we can write

$$3u - 3\int \frac{1}{u^3+1}\ \mathrm{du} = 3u - 3\int \frac{1}{(u+1)(u^2-u+1)}\ \mathrm{du}.$$

We proceed through partial fractional decomposition such that

$$3u - 3\int \frac{1}{(u+1)(u^2-u+1)}\ \mathrm{du} = 3u - 3\int \left(\frac{A}{u+1} + \frac{Bu+C}{u^2-u+1}\right)\ \mathrm{du}.$$

Evaluation for A, B, and C follows.

$$\frac{1}{(u+1)(u^2-u+1)} = \frac{A}{u+1} + \frac{Bu+C}{u^2-u+1}$$

$$1 = Au^2 - Au + A + Bu^2 + Cu + Bu + C$$

$$1 = (A+B)u^2 + (B+C-A)u + (A+C).$$

Thus, $A+B=0$, $B+C-A=0$, and $A+C=1$, which can be solved as a system of equations. Calculating accordingly shows that $A=\frac{1}{3}$, $B=-\frac{1}{3}$, and $C=\frac{2}{3}$. Therefore,

$$3u - 3\int \frac{1}{(u+1)(u^2-u+1)}\,\mathrm{du} = 3u - 3\int \left(\frac{\frac{1}{3}}{u+1} + \frac{-\frac{1}{3}u+\frac{2}{3}}{u^2-u+1}\right)\mathrm{du} = 3u - \int \left(\frac{1}{u+1} + \frac{2-u}{u^2-u+1}\right)\mathrm{du}.$$

Rearranging some terms allows

$$3u - \int \left(\frac{1}{u+1} + \frac{2-u}{u^2-u+1}\right)\mathrm{du} = 3u - \int \frac{1}{u+1}\,\mathrm{du} + \int \frac{u-2}{u^2-u+1}\,\mathrm{du}.$$

While the former of these integrands has a trivial resultant, we need to manipulate the numerator of the latter into a better form, as shown.

$$3u - \int \frac{1}{u+1}\,\mathrm{du} + \int \frac{u-2}{u^2-u+1}\,\mathrm{du} = 3u - \ln|u+1| + \frac{1}{2}\int \frac{2u-4}{u^2-u+1}\,\mathrm{du}.$$

Then, we separate -4 in the numerator of the integrand into $-1-3$, yielding

$$3u - \ln|u+1| + \frac{1}{2}\int \frac{2u-1-3}{u^2-u+1}\,\mathrm{du} = 3u - \ln|u+1| + \frac{1}{2}\int \frac{2u-1}{u^2-u+1}\,\mathrm{du} - \frac{3}{2}\int \frac{1}{u^2-u+1}\,\mathrm{du}.$$

The former of these integrand is once again trivial, but for the latter we require a manipulation of the denominator, as depicted.

$$3u - \ln|u+1| + \frac{1}{2}\ln|u^2-u+1| - \frac{3}{2}\int \frac{1}{\left(u-\frac{1}{2}\right)^2+\frac{3}{4}}\,\mathrm{du}.$$

The integrand herein is of an arctangent resultant form, giving

$$3u - \ln|u+1| + \frac{1}{2}\ln|u^2-u+1| - \frac{3}{2}\frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}\left(u-\frac{1}{2}\right)\right) + C$$

as an equivalent expression. Simplifying slightly and reintroducing $u=\sqrt[3]{x}$ leaves us with

$$\boxed{3\sqrt[3]{x} - \ln|\sqrt[3]{x}+1| + \frac{1}{2}\ln\left|x^{\frac{2}{3}} - \sqrt[3]{x}+1\right| - \sqrt{3}\arctan\left(\frac{2}{\sqrt{3}}\sqrt[3]{x} - \frac{1}{\sqrt{3}}\right) + C}.$$

Problem 3. $\displaystyle\int \frac{2x-9\sqrt{x}+9}{(x-3\sqrt{x})^{1/3}}\,\mathrm{dx}$

Solution: $\boxed{\dfrac{6}{5}(x-3\sqrt{x})^{5/3}+C}$

To remove the square root, we consider a u-substitution of $u=\sqrt{x}$, which implies that $\mathrm{du} = 1/(2\sqrt{x})\,\mathrm{dx}$. Therefore, we obtain the equivalent form of the integral as follows.

$$\int \frac{2x-9\sqrt{x}+9}{(x-3\sqrt{x})^{1/3}}\,\mathrm{dx} = \int \frac{2x-9u+9}{(x-3u)^{1/3}}\ (2\sqrt{x}\,\mathrm{du})$$

However, we still have a mixed integrand in terms of u and x. To solve this issue, we can solve for $x = u^2$, and use our initial u-substitution to make our integrand solely in terms of u as follows.

$$\int \frac{2x-9u+9}{(x-3u)^{1/3}}\left(2\sqrt{x}\ \mathrm{du}\right) = \int \frac{2u^2-9u+9}{(u^2-3u)^{1/3}}\left(2u\ \mathrm{du}\right) = \int \frac{4u^3-18u^2+18u}{(u^2-3u)^{1/3}}\ \mathrm{du}$$

We can now induce the denominator in the numerator by adding and subtracting $6u^2$ as follows.

$$\int \frac{4u^3-18u^2+6u^2-6u^2+18u}{(u^2-3u)^{1/3}}\ \mathrm{du} = \int \frac{(4u^3-12u^2)-(6u^2-18u)}{(u^2-3u)^{1/3}}\ \mathrm{du} = \int \frac{4u(u^2-3u)-6(u^2-3u)}{(u^2-3u)^{1/3}}\ \mathrm{du}$$

Simplifying, we have the following.

$$\int \frac{4u(u^2-3u)-6(u^2-3u)}{(u^2-3u)^{1/3}}\ \mathrm{du} = \int \frac{(4u-6)(u^2-3u)}{(u^2-3u)^{1/3}}\ \mathrm{du} = \int 2(2u-3)(u^2-3u)^{2/3}\ \mathrm{du}$$

Finally, we recognize that $2u-3$ is simply the derivative of u^2-3u, which allows us to claim that $v = u^2-3u$ and implying that $\mathrm{dv} = 2u-3\ \mathrm{du}$.

$$\int 2(2u-3)(u^2-3u)^{2/3}\ \mathrm{du} = \int 2v^{2/3}\ \mathrm{dv} = 2\frac{v^{\frac{2}{3}+1}}{\frac{2}{3}+1}+C = \frac{6}{5}v^{5/3}+C$$

From this, we invert our initial u-substitution to recover the initial solution in terms of x. Therefore, we can determine the equivalent form of the integral as follows.

$$\frac{6}{5}v^{5/3}+C = \frac{6}{5}(u^2-3u)^{5/3}+C = \boxed{\frac{6}{5}(x-3\sqrt{x})^{5/3}+C}$$

Problem 4. $\displaystyle\int_0^\infty \frac{\sin\sqrt{3}x - \sin\frac{x}{\sqrt{3}}}{x}e^{-x^2}\ \mathrm{dx}$

Solution: $\boxed{\dfrac{\pi}{6}}$

The form of the sine terms in the integral hint that we may want to start by considering the reverse of Feynman's trick, where rather than differentiating the integral with respect to a parameter, we re-express one of the terms in the integral as another integral. In this case, we notice that

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} x\cos(xt)\ \mathrm{dt} = \sin\sqrt{3}x - \sin\frac{x}{\sqrt{3}}$$

Where we treat x as a constant. Then,

$$\int_0^\infty \frac{e^{-x}}{x}\big(\sin\sqrt{3}x - \sin\frac{x}{\sqrt{3}}\big) = \int_0^\infty \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{e^{-x}}{x}x\cos(xt)\ \mathrm{dt}\ \mathrm{dx}$$

Letting the x's cancel and reversing the order of integration, we are left with a standard by-parts inner integral:

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \int_0^\infty e^{-x}\cos(xt)\ \mathrm{dx}\ \mathrm{dt}$$

By doing integration by-parts twice with respect to x, we will find that our inner integral evaluates to $\frac{1}{1+t^2}$. So now we are left with a single integral:

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+t^2}\ \mathrm{dt}$$

Which is another standard integral, which evaluates to $\arctan(t)\Big]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} = \frac{\pi}{3} - \frac{\pi}{6} = \boxed{\frac{\pi}{6}}$.

Problem 5. $\displaystyle\int_0^1 \frac{x^3}{(3+2x^2)^2}\ \mathrm{dx}$

Solution: $\boxed{\frac{1}{8}\ln\left(\frac{5}{3}\right) - \frac{1}{20}}$

If we try to expand the brackets we'll quickly see that the integral gets rather messy and doesn't reveal any nice way of solving it. So instead, consider the familiar integral

$$\int_0^1 \frac{x}{ax^2+b}\ \mathrm{dx}$$

Using the substitution $u = ax^2+b$, we find that the integral evaluates to

$$\int_0^1 \frac{x}{ax^2+b}\ \mathrm{dx} = \frac{1}{2a}\ln\left(1+\frac{a}{b}\right)$$

Let us then differentiate both sides of the equation with respect to a. When taking the derivative of an integral, it is equivalent to taking the partial derivative of the integrand with respect to the desired variable. So,

$$\int_0^1 \frac{-x^3}{(ax^2+b)^2} = -\frac{1}{2a^2}\ln\left(1+\frac{a}{b}\right) + \frac{1}{2a(a+b)}$$

Which is exactly the negative of our integral for the case $a = 2.b = 3$. So, substituting in numbers we obtain,

$$\int_0^1 \frac{x^3}{(3+2x^2)^2} = \boxed{\frac{1}{8}\ln\left(\frac{5}{3}\right) - \frac{1}{20}}$$

Problem 6. $\displaystyle\int_0^1 \arctan(x^2-x+1)\ \mathrm{dx}$

Solution: $\boxed{\ln(2)}$

Let us label our integral I. Recall the angle addition formula: $\tan(a+b) = \frac{\tan a + \tan b}{1-\tan a \tan b}$ and consider the function f:

$$f = \arctan\frac{1}{x} + \arctan(x-1)$$

Then,

$$\tan f = \tan(\arctan\frac{1}{x} + \arctan(x-1))$$

Using the tangent addition formula, we find

$$\tan(f) = x^2 - x + 1$$

Hence,

$$f = \arctan(x^2 - x + 1)$$

So we see that f is equivalent to the integrand of I which means that we can write

$$I = \int_0^1 \arctan\frac{1}{x}\,\mathrm{dx} + \int_0^1 \arctan x - 1\,\mathrm{dx}$$

Applying the identity $\arctan\frac{1}{x} = \frac{\pi}{2} - \arctan(x)$, for all $x > 0$ to the first integral, and the substitution $u = x - 1$ to the second integral, we find that

$$I = \frac{\pi}{2} - 2\int_0^1 \arctan(x)\,\mathrm{dx}$$

This can be evaluated by parts by setting $f(x) = \arctan(x)$ and $g'(x) = 1$ to obtain

$$I = \frac{\pi}{2} - 2\left[x\arctan(x) - \frac{1}{2}\ln(1+x^2)\right]_0^1$$

Which, when evaluated, yields $\boxed{I = \ln(2)}$.

Problem 7. $\displaystyle\int_0^\infty e^{-x^2}\cos(2x)\,\mathrm{dx}$

Solution: $\boxed{\dfrac{\sqrt{\pi}}{2e}}$

Consider an integral I as a function of some arbitrary parameter β such that

$$I(\beta) = \int_0^\infty e^{-x^2}\cos(\beta x)\,\mathrm{dx},$$

where $I(2)$ is clearly our goal. Differentiating I with respect to β,

$$\frac{\mathrm{d}I}{\mathrm{d}\beta} = \frac{\partial}{\partial\beta}\int_0^\infty e^{-x^2}\cos(\beta x)\,\mathrm{dx} = \int_0^\infty \frac{\partial}{\partial\beta}e^{-x^2}\cos(\beta x)\,\mathrm{dx} = \frac{1}{2}\int_0^\infty -2xe^{-x^2}\sin(\beta x)\,\mathrm{dx};$$

note that the second equality is derived from the Leibniz integral rule for differentiation under the integral sign. Moreover notice that there is a clear opportunity for integration by parts on which we act, as follows.

$$\frac{\mathrm{d}I}{\mathrm{d}\beta} = \frac{1}{2}\left[e^{-x^2}\sin(\beta x)\Big|_0^\infty - \beta\int_0^\infty e^{-x^2}\cos(\beta x)\,\mathrm{dx}\right].$$

The first term within the brackets collapses, and we recognize the second term as $-I(\beta)$. Thus,

$$\frac{\mathrm{d}I}{\mathrm{d}\beta} = -\frac{\beta}{2}I(\beta),$$

which is generally solved through separation, yielding

$$I(\beta) = Ce^{-\frac{\beta^2}{4}}$$

for some $C \in \mathbb{R}$. We notice that

$$I(0) = C = \int_0^\infty e^{-x^2}\,\mathrm{dx} = \frac{\sqrt{\pi}}{2},$$

where the integral evaluation is deduced from the well-known Gaussian integral. Thus,

$$I(\beta) = \frac{\sqrt{\pi}}{2}e^{-\frac{\beta^2}{4}}$$

is a new expression for I strictly in terms of the parameter β. Fixing $\beta = 2$ allows

$$I(2) = \boxed{\frac{\sqrt{\pi}}{2e}},$$

as desired.

Problem 8. $\displaystyle\int \frac{1}{2+2\sin(x)+\cos(x)}\,\mathrm{dx}$

Solution: $\boxed{\ln\left(\frac{\tan\left(\frac{x}{2}\right)+1}{\tan\left(\frac{x}{2}\right)+3}\right)+C}$

With nothing but trigonometric functions present as motivation for a method of integration, we may as well utilize a Weierstrass substitution such that

$$t = \tan\left(\frac{x}{2}\right) \implies \mathrm{dx} = \frac{2}{1+t^2}\,\mathrm{dt}.$$

This statement may be applied to trigonometric identities of the sine and cosine function listed below.

$$\sin(x) = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} = \frac{2t}{1+t^2};$$
$$\cos(x) = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} = \frac{1-t^2}{1+t^2}.$$

We are thus prepared to make our substitution:

$$\int \frac{1}{2+2\sin(x)+\cos(x)}\,\mathrm{dx} = \int \frac{\frac{2}{1+t^2}}{2+\frac{4t}{1+t^2}+\frac{1-t^2}{1+t^2}}\,\mathrm{dt} = \int \frac{2}{t^2+4t+3}\,\mathrm{dt} = \int \frac{2}{(t+1)(t+3)}\,\mathrm{dt}.$$

We now find it pertinent to decompose this integrand into partial fractions, which yields that

$$\int \frac{2}{(t+1)(t+3)}\,\mathrm{dt} = \int \left(\frac{1}{t+1} - \frac{1}{t+3}\right)\mathrm{dt} = \ln\left(\frac{t+1}{t+3}\right) + C = \boxed{\ln\left(\frac{\tan\left(\frac{x}{2}\right)+1}{\tan\left(\frac{x}{2}\right)+3}\right)+C}.$$

Problem 9. $\displaystyle\int \prod_{k=1}^{n-1} \sqrt{2x^2 - 2x^2\cos\left(\frac{2\pi k}{n}\right)}\,\mathrm{dx}$

Solution: $\boxed{x^n + C}$

We immediately notice a factorization may occur under the square root in the form of

$$\int \prod_{k=1}^{n-1} \sqrt{2x^2 - 2x^2\cos\left(\frac{2\pi k}{n}\right)}\,\mathrm{dx} = \int \prod_{k=1}^{n-1} x\sqrt{2}\sqrt{1-\cos\left(\frac{2\pi k}{n}\right)}\,\mathrm{dx}.$$

Now, the $x\sqrt{2}$ factor is being multiplied with itself $n-1$ times as per the upper bound of the product. This coincides with the definition of exponentiation, allowing us to reduce our integral to

$$\int \prod_{k=1}^{n-1} x\sqrt{2}\sqrt{1-\cos\left(\frac{2\pi k}{n}\right)}\,\mathrm{dx} = \int (x\sqrt{2})^{n-1} \prod_{k=1}^{n-1} \sqrt{1-\cos\left(\frac{2\pi k}{n}\right)}\,\mathrm{dx} = \int x^{n-1} 2^{\frac{n-1}{2}} \prod_{k=1}^{n-1} \sqrt{1-\cos\left(\frac{2\pi k}{n}\right)}\,\mathrm{dx}.$$

Let us deviate for a moment and consider the constant factor with respect to x,

$$2^{\frac{n-1}{2}} \prod_{k=1}^{n-1} \sqrt{1-\cos\left(\frac{2\pi k}{n}\right)},$$

and analyze what it might equal for increasing values of n. We notice $n \in \mathbb{N}_{\geq 2}$, so fix $n = 2$; then,

$$2^{\frac{2-1}{2}} \prod_{k=1}^{2-1} \sqrt{1-\cos\left(\frac{2\pi k}{2}\right)} = \sqrt{2}\sqrt{2} = 2.$$

Fix $n = 3$; then,

$$2^{\frac{3-1}{2}} \prod_{k=1}^{3-1} \sqrt{1-\cos\left(\frac{2\pi k}{3}\right)} = 2\sqrt{\frac{3}{2}}\sqrt{\frac{3}{2}} = 3.$$

Fix $n = 4$; then,

$$2^{\frac{4-1}{2}} \prod_{k=1}^{4-1} \sqrt{1-\cos\left(\frac{2\pi k}{4}\right)} = 2\sqrt{2}1\sqrt{2}1 = 4.$$

Interestingly, we notice that for this small collection of fixed values of n,

$$n = 2^{\frac{n-1}{2}} \prod_{k=1}^{n-1} \sqrt{1-\cos\left(\frac{2\pi k}{n}\right)}.$$

One may then perform mathematical induction on this statement to test whether it holds generally, which it indeed does. We may then make a substitution utilizing this statement, yielding

$$\int x^{n-1} 2^{\frac{n-1}{2}} \prod_{k=1}^{n-1} \sqrt{1-\cos\left(\frac{2\pi k}{n}\right)}\,\mathrm{dx} = \int n x^{n-1}\,\mathrm{dx} = \boxed{x^n + C}.$$

Problem 10. $\displaystyle\int_0^\infty e^{-x}\sin(x)\ln(x)\ \mathrm{dx}$

Solution: $\boxed{\dfrac{\pi}{8}-\dfrac{\ln(2)}{4}-\dfrac{\gamma}{2}}$

Notice

$$\int_0^\infty e^{-x}\sin(x)\ln(x)\ \mathrm{dx}=\int_0^\infty e^{-x}\Im\left(e^{ix}\right)\ln(x)\ \mathrm{dx}=\Im\left(\int_0^\infty e^{(i-1)x}\ln(x)\ \mathrm{dx}\right),$$

and, for simplicity, let

$$\mathscr{I}=\int_0^\infty e^{(i-1)x}\ln(x)\ \mathrm{dx}.$$

Utilizing the definition

$$\lim_{N\to\infty}\left(1+\frac{(i-1)x}{N}\right)^{N-1}=e^{(i-1)x},$$

it follows that

$$\int_0^\infty e^{(i-1)x}\ln(x)\ \mathrm{dx}=\lim_{N\to\infty}\int_0^N\left(1+\frac{(i-1)x}{N}\right)^{N-1}\ln(x)\ \mathrm{dx}.$$

Performing a change of variables with

$$z=1+\frac{(i-1)x}{N}$$

yields

$$\mathscr{I}=\lim_{N\to\infty}\frac{N}{i-1}\int_1^i z^{N-1}\ln\left(\frac{N(1-z)}{1-i}\right)\ \mathrm{dz}$$

as an equivalent expression. This is then equal to

$$\lim_{N\to\infty}\left(\frac{N(\ln(N)-\ln(1-i))}{i-1}\int_1^i z^{N-1}\ \mathrm{dz}+\frac{N}{i-1}\int_1^i z^{N-1}\ln(1-z)\ \mathrm{dz}\right)$$

through properties of logarithms. Evaluating the former of these integrals gives

$$\mathscr{I}=\lim_{N\to\infty}\left(\frac{\left(i^N-1\right)\left(\ln(N)-\ln(1-i)\right)}{i-1}+\frac{N}{i-1}\int_1^i z^{N-1}\ln(1-z)\ \mathrm{dz}\right).$$

Moreover, since

$$\ln(1-z)=-\sum_{k=1}^{\infty}\frac{z^k}{k},$$

$$\mathscr{I}=\lim_{N\to\infty}\left(\frac{\left(i^N-1\right)\left(\ln(N)-\ln(1-i)\right)}{i-1}-\frac{N}{i-1}\int_1^i\sum_{k=1}^{N}\frac{z^{N+k-1}}{k}\ \mathrm{dz}\right).$$

By the dominated convergence theorem, the discrete summation and integral can be interchanged; thus,

$$\mathscr{I}=\lim_{N\to\infty}\left(\frac{\left(i^N-1\right)\left(\ln(N)-\ln(1-i)\right)}{i-1}-\frac{N}{i-1}\sum_{k=1}^{N}\int_1^i\frac{z^{N+k-1}}{k}\ \mathrm{dz}\right),$$

meaning that

$$\mathscr{I} = \lim_{N\to\infty}\left(\frac{\left(i^N-1\right)\left(\ln(N)-\ln(1-i)\right)}{i-1} - \frac{N}{i-1}\sum_{k=1}^{N}\frac{i^{N+k}-1}{k(N+k)}\right).$$

Consider the following:

$$\begin{aligned}\lim_{N\to\infty}\frac{N}{i-1}\sum_{k=1}^{N}\frac{i^{N+k}-1}{k(N+k)} &= \lim_{N\to\infty}\left(\frac{Ni^N}{i-1}\sum_{k=1}^{N}\frac{i^k}{k(N+k)} - \frac{N}{i-1}\sum_{k=1}^{N}\frac{1}{k(N+k)}\right)\\ &= \lim_{N\to\infty}\left(\frac{i^N}{i-1}\sum_{k=1}^{N}\left(\frac{i^k}{k}-\frac{i^k}{N+k}\right) - \frac{1}{i-1}\sum_{k=1}^{N}\left(\frac{1}{k}-\frac{1}{N+k}\right)\right)\\ &= \lim_{N\to\infty}\left(\frac{i^N}{i-1}\sum_{k=1}^{N}\frac{i^k}{k} - \frac{1}{i-1}\sum_{k=1}^{N}\frac{1}{k}\right) = \lim_{N\to\infty}\left(-\frac{i^N\ln(1-i)}{i-1} - \frac{1}{i-1}\sum_{k=1}^{N}\frac{1}{k}\right).\end{aligned}$$

Making a substitution,

$$\mathscr{I} = \lim_{N\to\infty}\left(\frac{\left(i^N-1\right)\left(\ln(N)-\ln(1-i)\right)}{i-1} + \frac{i^N\ln(1-i)}{i-1} + \frac{1}{i-1}\sum_{k=1}^{N}\frac{1}{k}\right).$$

Redistribution yields

$$\mathscr{I} = \frac{1}{i-1}\lim_{N\to\infty}\left(i^N\ln(N) + \ln(1-i) + \sum_{k=1}^{N}\frac{1}{k} - \ln(N)\right).$$

Recognizing the presence of the Euler-Mascheroni constant and representing $1-i$ exponentially leads to

$$\mathscr{I} = \frac{1}{i-1}\left(\lim_{N\to\infty} i^N\ln(N) + \ln\left(\sqrt{2}e^{-\frac{i\pi}{4}}\right) + \gamma\right).$$

Again using logarithmic properties,

$$\mathscr{I} = \frac{1}{i-1}\left(\lim_{N\to\infty} i^N\ln(N) + \frac{\ln(2)}{2} - \frac{i\pi}{4} + \gamma\right),$$

which can be simplified to

$$-\frac{i+1}{2}\lim_{N\to\infty} i^N\ln(N) - \frac{i\ln(2)}{4} - \frac{\ln(2)}{4} - \frac{\pi}{8} + \frac{i\pi}{8} - \frac{i\gamma}{2} - \frac{\gamma}{2}.$$

Recall that

$$\int_0^\infty e^{-x}\sin(x)\ln(x)\ \mathrm{dx} = \Im(\mathscr{I}).$$

Thus,

$$\int_0^\infty e^{-x}\sin(x)\ln(x)\ \mathrm{dx} = \frac{\pi}{8} - \frac{\ln(2)}{4} - \frac{\gamma}{2} - \frac{1}{2}\Im\left((i+1)\lim_{N\to\infty} i^N\ln(N)\right).$$

Consider the following:

$$\begin{aligned}&= (i+1)\lim_{N\to\infty} i^N\ln(N) = \lim_{N\to\infty} i^{N+1}\ln(N) + \lim_{N\to\infty} i^N\ln(N)\\ &= \lim_{N\to\infty} i^{N+2}\ln(N) + \lim_{N\to\infty} i^N\ln(N) = \lim_{N\to\infty} i^N\ln(N) - \lim_{N\to\infty} i^N\ln(N) = 0.\end{aligned}$$

Therefore,

$$\int_0^\infty e^{-x}\sin(x)\ln(x)\ \mathrm{dx} = \boxed{\frac{\pi}{8} - \frac{\ln(2)}{4} - \frac{\gamma}{2}}.$$