Team Round 2022-2023 Solutions

Problem 1. A wall contains three switches A,B,C, each of which powers a light when flipped on. Every 20 seconds, switch A is turned on and then immediately turned off again. The same occurs for switch B every 21 seconds and switch C every 22 seconds. At time $t = 0$, all three switches are simultaneously on.

Let $t = T > 0$ be the earliest time that all three switches are once again simultaneously on. Compute the number of times $t > 0$ before T when at least two switches are simultaneously on.

Proposed by Nathan Hasegawa

Solution: $\boxed{39}$.

Since switches A,B,C are on only when t is a multiple of $20,21,22$, respectively, the earliest time T that all three switches are once again on is precisely $\text{lcm}(20,21,22)$. Now, switches A,B are simultaneously on only when t is a common multiple of 20 and 21, i.e. when t is a multiple of $\text{lcm}(20,21) = 420$. The number of positive multiples of $\text{lcm}(20,21)$ less than $\text{lcm}(20,21,22)$, i.e. the number of times $t > 0$ switches A,B are simultaneously on before time T, is exactly $\frac{\text{lcm}(20,21,22)}{\text{lcm}(20,21)} - 1$. The analagous statements hold for switches B,C and switches A,C. Moreover, for any time $0 < t < T$ when at least two switches are on, there are in fact exactly two switches on. Compute $\text{lcm}(20,21) = 420, \text{lcm}(21,22) = 462, \text{lcm}(20,22) = 220, \text{lcm}(20,21,22) = 4620$. Hence, the answer is

$$\left(\frac{4620}{420} - 1\right) + \left(\frac{4620}{462} - 1\right) + \left(\frac{4620}{220} - 1\right) = 10 + 9 + 20 = \boxed{39}.$$

Problem 2. Select a number X from the set of all 3-digit natural numbers uniformly at random. Let $A \in [0,1]$ be the probability that X is divisible by 11, given that it is palindromic. Let $B \in [0,1]$ be the probability that X is palindromic, given that it is divisible by 11. Compute $B - A$.

Recall that a 3-digit number is a palindrome if it reads the same left to right as right to left. For instance, 484 *is a palindrome, but* 603 *is not a palindrome.*

Proposed by Natalie Couch

Solution: $\boxed{\frac{4}{405}}$.

For any palindromic 3-digit number, there are 9 choices for the hundreds digit (which determines the units digit) and 10 choices for the tens digit, so there are a total of 90 palindromic 3-digit numbers. The smallest and largest 3-digit multiples of 11 are 110 and 990, respectively, so there are $\frac{1}{11} \cdot (990 - 110) + 1 = 81$ 3-digit multiples of 11. Now, a 3-digit number $\overline{abc}$ is both palindromic and a multiple of 11 if and only if $a = c$ and $a - b + c$ is a multiple of 11. Here, $a - b + c$ may only take the values 0 or 11 if it is a multiple of 11. Then, there are exactly 8 pairs (a,b) giving rise to 3-digit palindromic multiples of 11:

$$(1,2),(2,4),(3,6),(4,8),(6,1),(7,3),(8,5),(9,7).$$

This means $A = \frac{8}{90}$, $B = \frac{8}{81}$, so the answer is $B - A = \frac{8}{81} - \frac{8}{90} = \boxed{\frac{4}{405}}$.

Problem 3. Let $a_1, a_2, \ldots$ be a strictly increasing sequence of positive real numbers such that $a_1 = 1, a_2 = 4$, and that for every positive integer k, the subsequence $a_{4k-3}, a_{4k-2}, a_{4k-1}, a_{4k}$ is geometric and the subsequence $a_{4k-1}, a_{4k}, a_{4k+1}, a_{4k+2}$ is arithmetic. For each positive integer k, let r_k be the common ratio of the geoemtric sequence $a_{4k-3}, a_{4k-2}, a_{4k-1}, a_{4k}$. Compute

$$\sum_{k=1}^{\infty}(r_k-1)(r_{k+1}-1).$$

Proposed by Brian Yang

Solution: $\boxed{\frac{3}{2}}$.

For each positive integer k, we may write $a_{4k+1} = 2a_{4k} - a_{4k-1},\ a_{4k+2} = 3a_{4k} - 2a_{4k-1}$ using the arithmetic sequence formula. Hence,

$$r_k = \frac{a_{4k}}{a_{4k-1}},\ r_{k+1} = \frac{a_{4k+2}}{a_{4k+1}} = \frac{3a_{4k}-2a_{4k-1}}{2a_{4k}-a_{4k-1}} = \frac{3-\frac{2}{r_k}}{2-\frac{1}{r_k}} = \frac{3r_k-2}{2r_k-1}$$

We claim $r_k = \frac{6(k-1)+4}{6(k-1)+1}$, with the proof by induction. The base case $k=1$ is clear. Given that the formula for r_k holds, we have

$$r_{k+1} = \frac{3\left(\frac{6(k-1)+4}{6(k-1)+1}\right)-2}{2\left(\frac{6(k-1)+4}{6(k-1)+1}\right)-1} = \frac{18(k-1)+12-12(k-1)-2}{12(k-1)+8-6(k-1)-1} = \frac{6k+4}{6k+1}.$$

which is exactly what we want. Hence, $r_k - 1 = \frac{3}{6(k-1)+1}$, so we want to compute

$$\frac{3}{1}\cdot\frac{3}{7}+\frac{3}{7}\cdot\frac{3}{13}+\cdots = 9\left(\frac{1}{1\cdot 7}+\frac{1}{7\cdot 13}+\dots\right).$$

For each positive integer k, we have the partial fraction decomposition $\frac{1}{(6k-5)(6k+1)} = \frac{1}{6}\cdot\left(\frac{1}{6k-5}-\frac{1}{6k+1}\right)$. So the above expression telescopes:

$$9\left(\frac{1}{1\cdot 7}+\frac{1}{7\cdot 13}+\dots\right) = 9\cdot\frac{1}{6}\cdot\left(\frac{1}{1}-\frac{1}{7}+\frac{1}{7}-\frac{1}{13}+\dots\right) = \boxed{\frac{3}{2}}.$$

Problem 4. Gus is an inhabitant on an 11 by 11 grid of squares. He can walk from one square to an adjacent square (vertically or horizontally) in 1 unit of time. There are also two vents on the grid, one at the top left and one at the bottom right. If Gus is at one vent, he can teleport to the other vent in 0.5 units of time. Let an ordered pair of squares (a,b) on the grid be *sus* if the fastest path from a to b requires Gus to teleport between vents. Walking on top of a vent does not count as teleporting between vents.

What is the total number of ordered pairs of squares that are *sus*? Note that the pairs (a_1,b_1) and (a_2,b_2) are considered distinct if and only if $a_1 \neq a_2$ or $b_1 \neq b_2$.

Proposed by Mathus Leungpathomaram

Solution: $\boxed{1430}$.

Let $[n] = \{1,2,\dots,n\}$. We solve the problem for general n by n lattices $[n]^2 \subset \mathbb{Z}^2$ with vents v_0, v_1 at $(1,n)$ (the top left) and $(n,1)$ (the bottom right), respectively. For any points $a = (x_0,y_0), b = (x_1,y_1) \in \mathbb{Z}^2$, let $d(a,b)$ be its usual taxicab distance, and let $\tilde{d}(a,b) := |(y_1-x_1)-(y_0-x_0)|$ be an adjusted taxicab distance (geometrically,

$\tilde{d}(a,b)$ is the “taxicab distance” between the line of slope 1 through a and the line of slope 1 through b). Note that $\tilde{d}(a,b) \le d(a,b)$. For any square $a \in [n]^2$, define

$$B := B(a) = \{b \in [n]^2 \mid \tilde{d}(a,b) \ge n\}.$$

We claim (a,b) is *sus* if and only if $b \in B$.

Fix $a = (x_0, y_0) \in [n]^2$. For any $b = (x_1, y_1) \in [n]^2$, the fastest path from a to b is either a direct path (i.e., no teleporting between vents), taking $d(a,b)$ units of time, or teleports between vents at most once, taking $d(a,v_0) + d(v_1,b) + \frac{1}{2}$ or $d(a,v_1) + d(v_0,b) + \frac{1}{2}$ units of time. Now, observe $d(a,v_0) + d(v_1,b)$ is the taxicab distance from a to $(x_1 - (n-1), y_1 + (n-1))$, by applying a translation of $[n]^2$ that moves v_1 to v_0. Similarly, $d(a,v_1) + d(v_0,b)$ is the taxicab distance from a to $(x_1 + (n-1), y_1 - (n-1))$. Hence, (a,b) is *sus* if and only if $d(a,b) = |x_0 - x_1| + |y_0 - y_1|$ is strictly greater than at least one of

$$|x_0 - x_1 + (n-1)| + |y_0 - y_1 - (n-1)| = 2(n-1) + (x_0 - x_1) + (y_1 - y_0) = 2(n-1) \pm \tilde{d}(a,b),$$
$$|x_0 - x_1 - (n-1)| + |y_0 - y_1 + (n-1)| = 2(n-1) + (x_1 - x_0) + (y_0 - y_1) = 2(n-1) \mp \tilde{d}(a,b).$$

In particular, when $\tilde{d}(a,b) > n-1$, one of the above two expressions will be at most $n-1$, meaning (a,b) is *sus*.

In case $d(a,b) \le n-1$, we have $\tilde{d}(a,b) \le n-1$, in which the above condition shows that (a,b) is not *sus*. Now, suppose $n \le d(a,b) \le 2(n-1)$. Any shortest direct path from a to b will move $H = |x_1 - x_0|$ units horizontally and $V = |y_1 - y_0|$ units vertically; observe that $H, V \ge d(a,b) - (n-1)$. Then, $\tilde{d}(a,b) = |H \pm V|$. If we impose the condition $\tilde{d}(a,b) \le n-1$, then it is necessary that $\tilde{d}(a,b) = |H - V|$, meaning $\tilde{d}(a,b) \le 2(n-1) - d(a,b)$. Hence, (a,b) is not *sus* whenever $\tilde{d}(a,b) \le n-1$. This proves the desired claim on B.

For all integer values of d between 1 and $n-1$, there are $2d$ points a distance $d-1$ from a vent. For each of those $2d$ points a, our characterization of the region $B(a)$ gives rise to a simple geometric description: $B(a)$ is a right triangle—the right angle at the vent furthest from a—with base and height $n-d$ (hence consisting of $(n-d)(n-d+1)/2$ points). On the other hand, for any point a on the $x = y$ diagonal, the set $B(a)$ is empty. Hence, the total number of *sus* pairs is:

$$\sum_{d=1}^{n-1} d(n-d)(n-d+1).$$

Observe the value of the expression remains the same if we substitute d for $n-d$. Then, averaging the two quantities:

$$\begin{aligned}\sum_{d=1}^{n-1} d(n-d)(n-d+1) &= \frac{n+2}{2} \cdot \sum_{d=1}^{n-1} d(n-d) \\ &= \frac{n+2}{2} \cdot \frac{n(n-1)(n+1)}{6} = \frac{n(n+1)(n+2)(n-1)}{12}\end{aligned}$$

where we use the fact $\sum_{d=1}^{n-1} d(n-d) = \sum_{d=1}^{n-1} \sum_{k=1}^{d} k = \sum_{d=1}^{n-1} \binom{d+1}{2} = \binom{n+1}{3}$. Plugging in $n = 11$:

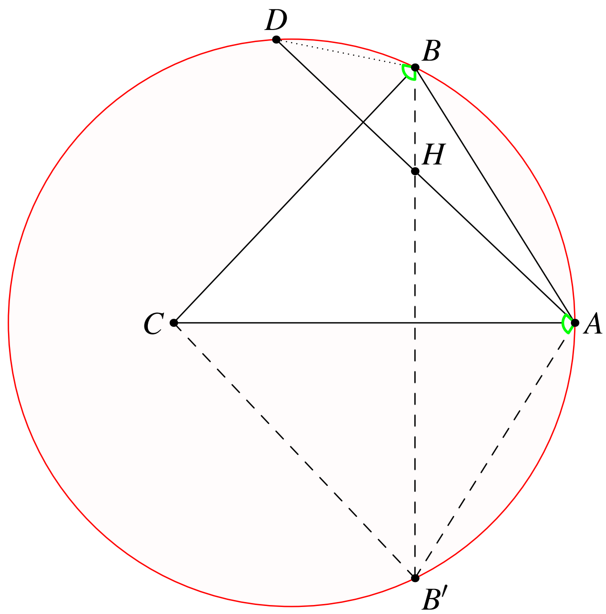
$$\frac{n(n+1)(n+2)(n-1)}{12} = \frac{13 \cdot 12 \cdot 11 \cdot 10}{12} = \boxed{1430}.$$

Problem 5. Let ABC be a triangle with $AB = 6, AC = 8, BC = 7$. Let H be the orthocenter of ABC. Let $D \ne H$ be a point on $\overline{AH}$ such that $\angle HBD = \frac{3}{2}\angle CAB + \frac{1}{2}\angle ABC - \frac{1}{2}\angle BCA$. Find DH.

Proposed by August Chen

Solution: $\boxed{\dfrac{251\sqrt{15}}{255}}$.

Denote by B' the reflection of B over side AC.



Note $\angle DBH = \frac{3}{2}\angle A + \frac{1}{2}\angle B - \frac{1}{2}\angle C = \angle A + \frac{1}{2}\cdot 180^\circ - \angle C = \angle A + 90^\circ - \angle C$. Notice that $\angle A > \angle C$, so $\angle DBH$ is obtuse. Thus, D must lie on $\overline{AH}$ such that D is closer to H than to A; otherwise, $\angle DHB = \angle AHB = 180^\circ - \angle C$, implying to the contrary $\angle DBH < \angle C$. So points A, B lie on the same side of $\overline{DD'}$. Moreover, $\angle HAB' = \angle HAC + \angle CAB' = 90^\circ - \angle C + \angle A$, so B', A, B, D are concyclic.

We want to compute HD using power of a point $HA \cdot HD = HB \cdot HB'$. Let R be the circumradius of $\triangle ABC$. By well-known properties of orthocenters, we have $HA = 2R\cos A, HB = 2R\cos B$. Now, $HB' = BB' - BH$, and $AC \cdot BB' = 4 \cdot [ABC] = 4 \cdot \frac{1}{2} \cdot (BC)(AB)\sin B$. We compute

$$\cos A = \frac{6^2 + 8^2 - 7^2}{2\cdot 6\cdot 8} = \frac{17}{32},\ \cos B = \frac{6^2+7^2-8^2}{2\cdot 6\cdot 7} = \frac{1}{4},\ \sin B = \frac{\sqrt{15}}{4},\ R = \frac{AC}{2\sin B} = \frac{16\sqrt{15}}{15}.$$

Putting everything together, we have

$$\begin{aligned} HD = \frac{HB}{HA}\cdot HB' &= \frac{\cos B}{\cos A}\cdot\left(2\cdot\frac{(BC)(AB)}{AC}\cdot\sin B - 2R\cos B\right) \\ &= \frac{1/4}{17/32}\cdot\left(2\cdot\frac{6\cdot 7}{8}\cdot\frac{\sqrt{15}}{4} - 2\cdot\frac{16\sqrt{15}}{15}\cdot\frac{1}{4}\right) = \boxed{\frac{251\sqrt{15}}{255}}. \end{aligned}$$

Problem 6. Let A be a set of 8 elements, and $\mathscr{B} := (B_1, \ldots, B_7)$ be an ordered 7-tuple of subsets of A. Let N be the number of such 7-tuples $\mathscr{B}$ such that there exists a unique 4-element subset $I \subseteq \{1, 2, \ldots, 7\}$ for which the intersection $\bigcap_{i\in I} B_i$ is nonempty. Find the remainder when N is divided by 67.

Proposed by Brian Yang

Solution: $\boxed{23}$.

Let's write $A = \{1,2,\ldots,8\}$. Consider a 7×8 array of bits (0's and 1's), where row i represents B_i and column j represents $j \in A$. Namely, entry (i,j) equals 1 if and only if $j \in B_i$. Then, the set of all possible 7-tuples $\mathscr{B}$ is in one-to-one correspondence with such arrays of bits. Observe that under this bijection, the task is to compute the number of 7×8 arrays where there exists a non-empty subset $J \subseteq A$ (we call this the set of *activated columns*) and a 4-element subset $I \subseteq \{1,2,\ldots,7\}$, such that

- For each column $j \in J$, the entry (i,j) equals 1 precisely when $i \in I$.
- For each column $j \notin J$, at most 4 entries of column j equals 1.

That is, a 7×8 array satisfies the above conditions, i.e. is *valid*, if and only if the original tuple $\mathscr{B}$ satisfies the conditions of the problem with $J = \bigcap_{i\in I} B_i$. Remark any 7×8 array satisfying the above conditions must be associated to a unique pair (I,J), so it suffices to count the number of valid arrays over all possible pairs (I,J).

Let us fix I. Given $J \subseteq A$, there are $|J|$ activated columns, each of which has one possible configuration of bits, and $8-|J|$ inactive columns, each of which has $\binom{7}{0}+\binom{7}{1}+\binom{7}{2}+\binom{7}{3} = 64$ possible configurations of bits. Hence, there are a total of $2^{6(8-|J|)}$ possible configurations of bits. Now, for any given cardinality k, there are $\binom{8}{k}$ choices of J such that $|J| = k$. It follows that there are

$$\binom{8}{1}2^{42}+\binom{8}{2}2^{36}+\cdots+\binom{8}{7}2^{6}+\binom{8}{8} = (1+2^6)^8 - 2^{48} = 65^8 - 64^8$$

valid arrays over all pairs (I,J) with this fixed I. Since there are $\binom{7}{4} = 35$ choices of I, we have

$$N = 35(65^8 - 64^8) \implies N \equiv \boxed{23} \pmod{67}.$$

Problem 7. Let $\mathbb{N}_0$ be the set of all non-negative integers. Let $f : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ be a function such that for all non-negative integers a,b:

$$f(a,b) = f(b,a),$$
$$f(a,0) = 0,$$
$$f(a+b,b) = f(a,b)+b.$$

Compute

$$\sum_{i=0}^{30}\sum_{j=0}^{2^i-1} f(2^i, j).$$

Proposed by Mathus Leungpathomaram

Solution: $\boxed{2^{61}-2^{35}}$.

We begin with the following characterization of f:

Lemma: $f(a,b) = a+b-\gcd(a,b)$ when a,b are not both 0 and $f(0,0) = 0$.

Proof: We use strong induction over $n = \max\{a,b\}$. When $n = 0$, $f(0,0) = 0$. Suppose that the closed form is correct for all $n \le k$. Consider the case where $n = k+1$. Then $\max\{a,b\} = k+1$ and WLOG let $a = k+1$. If $b = k+1$:

$$f(a,b) = f(k+1,k+1) = f(0,k+1)+k+1 = k+1,$$

$$k+1=(k+1)+(k+1)-\gcd(k+1,k+1)=a+b-\gcd(a,b).$$

If $b=0$:
$$f(a,b)=f(k+1,0)=0=(k+1)+0-\gcd(k+1,0)=a+b-\gcd(a,b).$$

Otherwise, $1\le b\le k$, so $a-b\le k$, so $\max\{b,a-b\}\le k$, so:

$$f(a,b)=f(a-b,b)+b=((a-b)+b-\gcd(a-b,b))+b=a+b-\gcd(a-b,b).$$

By the Euclidean algorithm, $\gcd(a-b,b)=\gcd(a,b)$. Then,

$$f(a,b)=a+b-\gcd(a-b,b)=a+b-\gcd(a,b).$$

By the induction hypothesis, the closed form is correct for all non-negative integer values of a,b. $\square$

Now, note that
$$\sum_{j=0}^{2^i-1}\gcd(2^i,j)=\sum_{n=0}^{i-1}2^n2^{i-1-n}=i\cdot 2^{i-1}.$$

Then,

$$\sum_{j=0}^{2^i-1}f(2^i,j)=\sum_{j=0}^{2^i-1}2^i+j-\gcd(2^i,j),$$
$$=2^i\cdot(2^i-1)+\frac{(2^i)(2^i-1)}{2}-i\cdot 2^{i-1}=\frac{3}{2}\left(2^{2i}-2^i\right)-i\cdot 2^{i-1}.$$

We compute the original expression:

$$\sum_{i=0}^{N}\sum_{j=0}^{2^i-1}f(2^i,j)=\sum_{i=0}^{N}\frac{3}{2}\left(2^{2i}-2^i\right)-i\cdot 2^{i-1}$$
$$=\left(2^{2N+1}-3\cdot 2^N+1\right)-\left((N-1)\cdot 2^N+1\right)=2^{2N+1}-(N+2)\cdot 2^N.$$

Plugging in $N=30$, we obtain $2^{2\cdot 30+1}-32\cdot 2^{30}=\boxed{2^{61}-2^{35}}$.

Problem 8. Suppose $a_3x^3-x^2+a_1x-7=0$ is a cubic polynomial in x whose roots α,β,γ are positive real numbers satisfying
$$\frac{225\alpha^2}{\alpha^2+7}=\frac{144\beta^2}{\beta^2+7}=\frac{100\gamma^2}{\gamma^2+7}.$$

Find a_1.

Proposed by Brian Yang

Solution: $\boxed{\frac{77}{15}}$.

By Vieta's, $7(\alpha+\beta+\gamma)=\alpha\beta\gamma$. Suppose $\alpha=\sqrt{7}\cdot\tan A,\beta=\sqrt{7}\cdot\tan B$, where $0<A,B<\frac{\pi}{2}$ are uniquely determined acute angles. Then, we have

$$\frac{\gamma}{\sqrt{7}}=-\frac{1}{\sqrt{7}}\cdot\frac{\alpha+\beta}{1-\frac{\alpha\beta}{7}}=-\frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan(A+B)=\tan C$$

where $C = \pi - A - B$. This implies A, B, C are the angles of an acute triangle. Now, if $(0,1) \ni t = \sin A$, then $\cos A = \sqrt{1-t^2}$, so we have $\frac{\alpha}{\sqrt{7}} = \frac{t}{\sqrt{1-t^2}}$. Solving for t, we get $t = \frac{\alpha}{\sqrt{7+\alpha^2}}$. Similar formulas hold for B, C. Therefore, by the law of sines, the condition

$$\frac{15\alpha}{\sqrt{\alpha^2+7}} = \frac{12\beta}{\sqrt{\beta^2+7}} = \frac{10\gamma}{\sqrt{\gamma^2+7}}$$

says that A, B, C are the angles of a 4-5-6 acute triangle (where the sides of $4, 5, 6$ are opposite A, B, C, respectively). Using the law of cosines and the fact $\cos^2\theta + \sin^2\theta = 1$, we compute

	A	B	C
cos	$\frac{3}{4}$	$\frac{9}{16}$	$\frac{1}{8}$
sin	$\frac{\sqrt{7}}{4}$	$\frac{5\sqrt{7}}{16}$	$\frac{3\sqrt{7}}{8}$
tan	$\frac{\sqrt{7}}{3}$	$\frac{5\sqrt{7}}{9}$	$3\sqrt{7}$

Hence, $\tan A \tan B \tan C = \tan A + \tan B + \tan C = \frac{35\sqrt{7}}{9}$. By Vieta's, we obtain $\frac{1}{a_3} = \sqrt{7} \cdot \frac{35\sqrt{7}}{9}$. Moreover, Vieta's also implies

$$\frac{a_1}{a_3} = \alpha\beta + \beta\gamma + \gamma\alpha = 7(\tan A \tan B + \tan B \tan C + \tan C \tan A) = 7 \cdot \frac{7 \cdot 77}{27}.$$

Putting everything together, we get

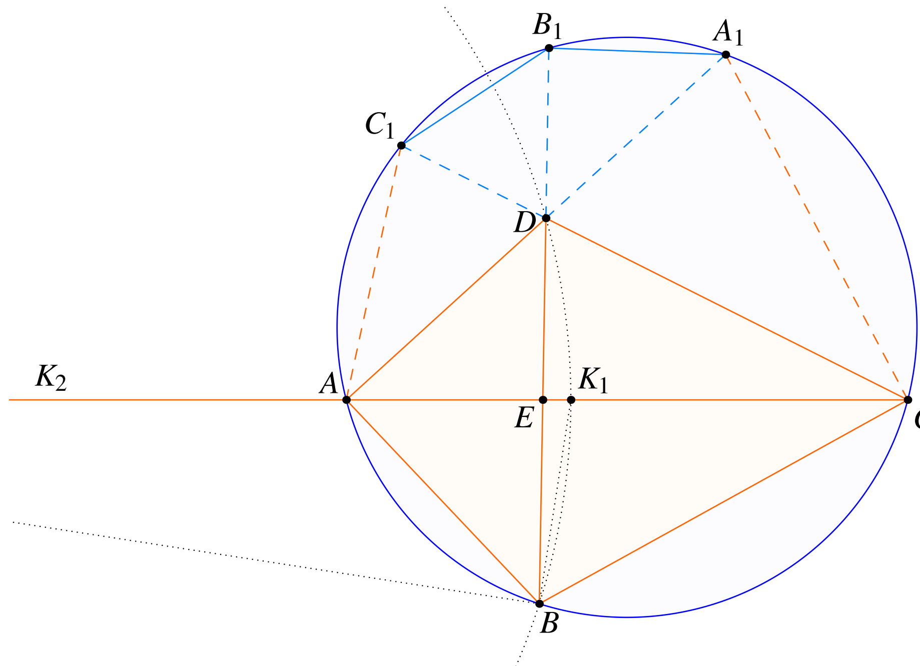
$$a_1 = 7 \cdot \frac{7 \cdot 77}{27} \cdot \frac{1}{\sqrt{7}} \cdot \frac{9}{35\sqrt{7}} = \boxed{\frac{77}{15}}.$$

Problem 9. Let $ABCD$ be a convex, non-cyclic quadrilateral with E the intersection of its diagonals. Given $\angle ABD + \angle DAC = \angle CBD + \angle DCA$, $AB = 10$, $BC = 15$, $AE = 7$, and $EC = 13$, find BD.

Proposed by Brian Yang

Solution: $\boxed{\frac{399\sqrt{211}}{422}}$.

Let $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$ intersect (ABC) again at A_1, B_1, and C_1, respectively.



Hence, there are pairs of similar triangles $\triangle ABD \sim \triangle B_1A_1D$, $\triangle BCD \sim \triangle C_1B_1D$, and $\triangle CA_1D \sim \triangle AC_1D$, i.e.

$$\frac{AB}{A_1B_1} = \frac{DA}{DB_1}, \frac{CA_1}{C_1A} = \frac{DC}{DA}, \frac{B_1C_1}{BC} = \frac{DB_1}{DC} \implies \frac{AB}{A_1B_1} \cdot \frac{CA_1}{C_1A} \cdot \frac{B_1C_1}{BC} = 1$$

The angle condition in the problem is equivalent to $\measuredangle DBA - \measuredangle DCA = \measuredangle CBD - \measuredangle CAD$. Observe

$$\measuredangle DBA - \measuredangle DCA = \measuredangle B_1BA - \measuredangle C_1CA = \measuredangle B_1CA - \measuredangle C_1CA = \measuredangle B_1CC_1$$
$$\measuredangle CBD - \measuredangle CAD = \measuredangle CBB_1 - \measuredangle CAA_1 = \measuredangle CAB_1 - \measuredangle C_1AA_1 = \measuredangle A_1AB_1,$$

so $A_1B_1 = B_1C_1$. Thus, $\frac{AB}{BC} = \frac{C_1A}{CA_1} = \frac{AD}{DC}$, i.e. D lies on the B-Apollonius circle of $\triangle ABC$, which we call Γ_B.
Denote by K_1 and K_2 the feet of the interior and exterior bisectors of $\angle ABC$ on $\overline{AC}$, respectively. It is well-known that $K_1, K_2 \in \Gamma_B$ (in fact $\overline{K_1K_2}$ is a diameter of Γ_B). By the angle bisector theorem, $K_2A = 40, AK_1 = 8$. Thus, $K_2E = K_2A + AE = 47, EK_1 = AK_1 - AE = 1$.
Now, by Stewart's Theorem,

$$AE \cdot EC \cdot AC + BE^2 \cdot AC = AB^2 \cdot EC + BC^2 \cdot EA \implies BE = \frac{\sqrt{211}}{2}.$$

By Power of a Point with respect to Γ_B, $BE \cdot ED = K_2E \cdot EK_1 = 47$. Thus, $ED = \frac{94}{\sqrt{211}}$, so it follows

$$BD = BE + ED = \frac{\sqrt{211}}{2} + \frac{94}{\sqrt{211}} = \boxed{\frac{399\sqrt{211}}{422}}.$$

Problem 10. Suppose that $\xi \neq 1$ is a root of the polynomial $f(x) = x^{167} - 1$. Compute

$$\left| \sum_{0<a<b<167} \xi^{a^2+b^2} \right|.$$

In the above summation a, b are integers.

Proposed by Brian Yang

Solution: $\boxed{3\sqrt{798}}$.

Let p be a prime with $p \equiv 7 \pmod 8$. Denote by $\mathbb{F}_p$ the field of order p (the set of all residues mod p) and $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$ its subset of invertible elements (the set of all non-zero residues mod p). Suppose that ξ is a primitive pth root of unity, and let $z = \sum_{0 \le a < p} \xi^{a^2}$.

Lemma 1: $|z| = \sqrt{p}$.

Proof 1: Note

$$z\bar{z} = \left(\sum_{0 \le a < p} \xi^{a^2}\right)\left(\sum_{0 \le a < p} \xi^{-a^2}\right) = \sum_{0 \le a,b < p} \xi^{a^2 - b^2} = \sum_{0 \le a,b < p} \xi^{(a-b)(a+b)}.$$

For any $(r,s) \in \mathbb{F}_p^2$, the system $a - b \equiv r \pmod p, a + b \equiv s \pmod p$ has the unique solution $a = \frac{r+s}{2}, b = \frac{r-s}{2}$ modulo p. In particular,

$$z\bar{z} = \sum_{0 \le a,b < p} \xi^{(a-b)(a+b)} = \sum_{0 \le r,s < p} \xi^{rs} = \sum_{r=0}^{p}\sum_{s=0}^{p} (\xi^r)^s.$$

When $r = 0$, $\sum_{s=0}^{p} (\xi^r)^s = \sum_{s=0}^{p} (1)^s = p$. When $r \neq 0$, ξ^r is a nonreal root of $x^p - 1$; hence, $\sum_{s=0}^{p} (\xi^r)^s$ is the sum of the roots of $x^p - 1$ which equals 0. Thus, $z\bar{z} = p \implies |z| = \sqrt{p}$. $\square$

Lemma 2: $z = \pm i\sqrt{p}$.

Proof 2: Denote by $Q \subset \mathbb{F}_p^*$ the subset of nonzero quadratic resides mod p. For any $t \in \mathbb{F}_p^*$, we have $-\left(\frac{t}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{t}{p}\right) = \left(\frac{-t}{p}\right)$, in lieu of the fact $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = -1$. In other words, exactly one of the two congruences $x^2 \equiv t \pmod p, x^2 \equiv -t \pmod p$ is solvable. Furthermore, any solvable quadratic congruence $x^2 \equiv t \pmod p$ has exactly two distinct solutions of the form $x \equiv \pm a \pmod p$ for some $a \in \mathbb{F}_p^*$. It follows $\sum_{0<a<p} \xi^{a^2} = 2 \cdot \sum_{u \in Q} \xi^u$, $\sum_{0<a<p} \xi^{-a^2} = 2 \cdot \sum_{u \in \mathbb{F}_p^* \setminus Q} \xi^u$. Hence,

$$z + \bar{z} = \sum_{0 \le a < p} \xi^{a^2} + \sum_{0 \le a < p} \xi^{-a^2} = 2 + 2 \cdot \sum_{u \in Q} \xi^u + 2 \cdot \sum_{u \in \mathbb{F}_p^* \setminus Q} \xi^u = 2 + 2 \cdot \sum_{u \in \mathbb{F}_p^*} \xi^u = 2 - 2 = 0.$$

Thus, z is pure imaginary, so $z = \pm i\sqrt{p}$. $\square$

Lemmas 1 and 2 derive the well-known basic properties of *quadratic Gauss sums*[1]. Now, we also wish to evaluate the sum $\sum_{0 \le a < p} \xi^{2a^2}$. The following Lemma 3 answers this question.

Lemma 3: $\sum_{0 \le a < p} \xi^{2a^2} = \sum_{0 \le a < p} \xi^{a^2}$.

Proof 3: Recall that $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = 1 \implies 2 \in Q$ (alternatively, in the special case $p = 167$, we have $(\pm 13)^2 \equiv 2 \pmod p$)). In particular, the permutation $\mathbb{F}_p^* \to \mathbb{F}_p^*, x \mapsto 2x$ acts as a permutation on $Q \subset \mathbb{F}_p^*$, implying the conclusion. $\square$

[1]Determining the sign of z requires the exact value of ξ and is more difficult. Here is a useful reference for further information on this topic: `https://www.mast.queensu.ca/~murty/quadratic2.pdf`.

Returning to the original problem:

$$z^2 = \left(\sum_{0\le a<p} \xi^{a^2}\right)^2 = \sum_{0\le a,b<p} \xi^{a^2+b^2} = \sum_{0\le a<p} \xi^{2a^2} + 2\cdot \sum_{0\le a<b<p} \xi^{a^2+b^2}$$
$$\implies \sum_{0\le a<b<p} \xi^{a^2+b^2} = \frac{z^2 - \sum_{0\le a<p} \xi^{2a^2}}{2} = \frac{-p \mp i\sqrt{p}}{2};$$
$$\sum_{0<a<b<p} \xi^{a^2+b^2} = \sum_{0\le a<b<p} \xi^{a^2+b^2} - \sum_{0<b<p} \xi^{b^2} = \sum_{0\le a<b<p} \xi^{a^2+b^2} - \left(\sum_{0\le b<p} \xi^{b^2} - 1\right)$$
$$\implies \sum_{0<a<b<p} \xi^{a^2+b^2} = \frac{-p \mp i\sqrt{p}}{2} \mp i\sqrt{p} + 1.$$

Hence,

$$\left|\sum_{0<a<b<p} \xi^{a^2+b^2}\right| = \sqrt{\left(\frac{p}{2} - 1\right)^2 + \left(\frac{3\sqrt{p}}{2}\right)^2} = \frac{\sqrt{(p+1)(p+4)}}{2}.$$

When $p = 167$, the above formula yields $\frac{\sqrt{168\cdot 171}}{2} = \boxed{3\sqrt{798}}$.