CHMMC 2015 Tiebreaker Problems

November 22, 2015

Problem 0.1. *Call a positive integer x n-cube-invariant if the last n digits of x are equal to the last n digits of x^3. For example, 1 is n-cube invariant for any integer n. How many 2015-cube-invariant numbers x are there such that $x < 10^{2015}$?*

Solution 1. $\boxed{14}$. *The equation $x \equiv x^3 \bmod 10^n$ implies $10^n | x(x-1)(x+1)$, and then chinese remainder theorem gives you 3 possible values mod 5^n but 5 possible values mod 2^n–in particular, the nontrivial ones mod 2^n are $2^{n-1} \pm 1$.*

Problem 0.2. *Let $a_1 = 1, a_2 = 1$, and for $n \geq 2$, let*

$$a_{n+1} = \frac{1}{n} a_n + a_{n-1}$$

What is a_{12}?

Solution 2. $\boxed{\frac{693}{256}}$. *This would normally be a very difficult problem, but trying out the first few terms gives $a_3 = 3/2, a_4 = 3/2, a_5 = 15/8, a_6 = 15/8$. If we suppose $a_{2k} = a_{2k-1}$, then we get*

$$a_{2k+1} = \frac{2k+1}{2k} a_{2k-1}$$

$$a_{2k+2} = \frac{1}{2k+1} a_{2k+1} + a_{2k-1} = \frac{2k+1}{2k} a_{2k-1} = a_{2k+1}$$

Which confirms that the trend holds forever. Also, we now get that $a_{12} = a_{11} = \frac{3*5*7*9*11}{2*4*6*8*10} =$ $\boxed{\frac{693}{256}}$

Problem 0.3. *Define an* n-digit pair cycle *to be a number with $n^2 + 1$ digits between 1 and n with every possible pair of consecutive digits. For instance, 11221 is a 2-digit pair cycle since it contains the consecutive digits 11, 12, 22, and 21. How many 3-digit pair cycles exist?*

Solution 3. $\boxed{216}$.

Every number from 1 to n must appear n times (excluding the last digit) in order for each consecutive pair to appear. The first time the digit appears, it can be followed by any of the n other numbers. The next time, there are only $n - 1$ choices, and so on. Therefore, for each number, there are $n!$ choices, independent of how the other numbers are chosen. Multiplying gives $(n!)^n$ total combinations, or $6^3 =$ **216** *in our case.*

Problem 0.4. *The following number is the product of the divisors of* n.

$$46,656,000,000$$

What is n*?*

Solution 4. $\boxed{60}$.

In general, the product of the divisors of n *is* $n^{\#\text{ of divisors of } n}$. 60 *has* 12 *divisors and* $60^6 = 46,656,000,000$.