CHMMC Power Round 2019 Solutions

November 17, 2019

1 Introduction

Solution 1. There are lots of possibilities. Any configuration which splits the square into n triangles of equal area, where n is an even positive integer, is correct. One possible answer (for $n = 10$ triangles) is which easily generalizes to any even n.

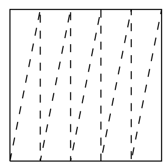

Figure 1: The square is split into $n = 10$ triangles of equal area. A similar configuration works for any even n.

Solution 2. Take the solution for some n. Each triangle has area $\frac{1}{n}$. Scale it vertically so that each triangle has area $\frac{1}{n+2}$. Then we are left with a rectangle with width 1 and height $\frac{n}{n+2}$. Finally add two more triangles, each of area $\frac{1}{n+2}$, to the top "missing" rectangle of width 1 and height $\frac{2}{n+2}$. See Figure 2.

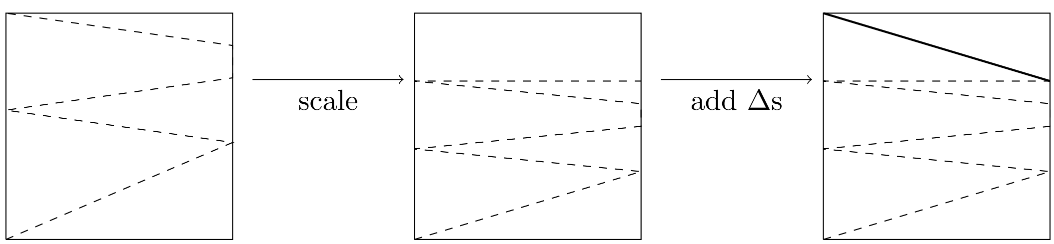

Figure 2: Figure for Solution 2. Start with a configuration with $n = 5$ triangles (not to scale since this is impossible), and construct a configuration with $n = 7$ triangles.

Solution 3. If (1) is not true, then consider that the minimal odd integer n' where it is possible to split a square into n' triangles of equal area. Write $n' = 2m + 1$. Any odd integer $2m' + 1$ where $m' > m$ is able to be split into $2m' + 1$ triangles of equal area by using the statement of Problem 2 $m' - m$ times. Hence the only positive odd integers n it might be not possible to split a square into n triangles of equal area are all less than n', so there are a finitely many number of such positive odd integers, so (2) is true.

2 The case $n = 3$

Solution 4. We use Euler's formula $V - E + F = 2$. If an n-gon is split in to k triangles, then the number of vertices is $V \geq n$ as we must have the vertices of the n-gon plus any added internal vertices, the number of edges is $2k + 1$ which we can show by induction , and the number of faces is $k+1$ as we need the k triangles plus 1 for the outside face. Plugging in these values gives $V = E - F + 2 = k + 2 \geq n$ from which the result follows.

Alternatively, proceed by induction on n. If $n = 3$ then we have a triangle. Hence it can not be split in to fewer than 1 triangles, so $k \geq 1 = n - 2$. Suppose that for $m \leq n$ an m-gon being split in to k triangles implies $k \geq m - 2$. Consider an $(n+1)$-gon. To minimize k the first edge should connect two vertices. This splits the $(n+1)$-gon in to an m-gon and an $(n-m+2)$-gon. These require an additional $m-2$ and $n-m$ edges to triangulate respectively by the induction hypothesis. Hence the total number of edges needed is at least $1 + (m-2) + (n-m) = n-1$ as desired.

Solution 5. Let the square be $ABCD$. Suppose by way of contradiction that there exists such a split.

- Following the hint, first suppose a triangle has a vertex P in the interior of the square. In order for this to form a triangle, WLOG the triangle has to be PAB. But then the square is split into at least four triangles, contradiction.

- Now suppose the triangles only have vertices on the sides and vertices of $ABCD$. Suppose there are two vertices on the sides of $ABCD$. Let Q be a vertex on the side AB, and let R be the second vertex on a side of $ABCD$. If R is on CD, then $QACR$ and $QBDR$ are both 4-gons, which means the square must be split into at least four triangles by Problem 4. If R is on an adjacent side, WLOG BC, then $QRCDA$ is a 5-gon. By Problem 4, the 5-gon must be split into at least 3 triangles, so the square must be split into at least 4 triangles. If R is on AB, then Q is connected to a vertex other than A and R is connected to a vertex other than B. Each creates a 4-gon, which means the square must be split into at least four triangles by Problem 4. Thus no matter what, we arrive at a contradiction.

- Thus Q is the only vertex on a side of $ABCD$. So QCD must be a triangle. This implies that $ABCD$ is split into ΔAQC, ΔABQ, and ΔBQD. Then because $\text{Area}(AQC) = \text{Area}(BQD)$, Q must be the midpoint of AB. But then $\text{Area}(ABQ) = 2\text{Area}(AQC)$, which is a contradiction.

3 The 2-adics

Solution 6. Write $n = \frac{2^m a}{2^n b}$ where $\gcd(a,b) = 1$, where a, b are both odd and where m, n are nonnegative integers. Then $-1 = v_2(n) = v_2(2^m a) - v_2(2^n b) = m - n$ so $n = m + 1 \implies n = \frac{a}{2b}$ for odd integers a, b with $\gcd(a,b) = 1$.

Solution 7. For any integers x, y write $x = 2^m a, y = 2^n b$ where a, b are odd integers. Remark $v_2(xy) = m + n = v_2(x) + v_2(y)$, $v_2(x) = m = v_2(-x)$ and that if $v_2(x) \neq v_2(y) \iff m \neq n$, $v_2(x+y) = v_2(2^m a + 2^n b) = \min(m,n) = \min(v_2(x), v_2(y))$. Now we use the definition of M_2.

(a) $M_2(xy) = 2^{-v_2(xy)} = 2^{-v_2(x) - v_2(y)} = 2^{-v_2(x)} 2^{-v_2(y)} = M_2(x) M_2(y)$.

(b) $M_2(x) = 2^{-v_2(x)} = 2^{-v_2(-x)} = M_2(-x)$.

(c) If $M_2(x) \neq M_2(y) \implies 2^{-v_2(x)} \neq 2^{-v_2(y)} \implies v_2(x) \neq v_2(y)$, then $v_2(x+y) = \min(v_2(x), v_2(y)) \implies -v_2(x+y) = \max(-v_2(x), -v_2(y))$, so $M_2(x+y) = 2^{-v_2(x+y)} = 2^{\max(-v_2(x), -v_2(y))} = \max(2^{-v_2(x)}, 2^{-v_2(y)}) = \max(M_2(x), M_2(y))$.

Solution 8. Let $x = a/b$ and $y = b/a$ for integers a, b where $\gcd(a, b) = 1$. If $M_2(x) > 1$, then $2^{-v_2(x)} > 1$, so $v_2(x) < 0$, so b is even and a is odd. Thus $M_2(y) < 1$. If y is an integer, then since b is even and a is odd, b/a is even.

4 Sperner's Lemma

Solution 9. Let K_1 be the number of dots inside the square. Note that every complete triangle has one dot inside it, since it has exactly one rg-edge. Moreover, if a triangle has one dot inside it, then it is complete, since the only way for a triangle to have one dot inside it is for it to have one rg-edge and the other vertex of the triangle to be blue. Hence, every non complete triangle has either 0 or 2 dots inside it. Thus $K_1 - C_1 = 0 \cdot \#$ triangles with 0 dots inside $+ \#$ triangles with 2 dots inside, so $K_1 - C_1$ is even.
The number of dots outside the square is C_2, since each dot outside the triangle is created due to a rg-edge on the boundary. Note that $K_1 = C_2 + 2 \cdot \#$ of interior rg-edges, so $C_2 - K_1$ is even.
Thus $-((C_2 - K_1) + (-C_1 + K_1)) = C_1 - C_2$ is also even.

5 Monsky's Theorem

Solution 10. Considering the x component, $S_1 \cap S_2 = S_1 \cap S_3 = \emptyset$. Considering the y component, $S_2 \cap S_3 = \emptyset$. Let (x, y) be an arbitrary point. If $(x, y) \notin S_1$, then $M_2(x) \geq 1$ and $M_2(y) \geq 1$. We must have either $M_2(x) \geq M_2(y)$ or $M_2(y) < M_2(x)$, which implies $(x, y) \in S_2$ or $(x, y) \in S_3$.

Solution 11. We use Problem 7.

(a) Using 7b) that $M_2(x) = M_2(-x)$ yields $M_2(-x_1) = M_2(x_1) < 1, M_2(-y_1) = M_2(y_1) < 1 \implies (-x_1, -y_1) \in S_1$.

(b) By 7c), First we have $M_2(x_1 + x_2) = \max(M_2(x_1), M_2(x_2)) = M_2(x_2) \geq 1$. Additionally, $M_2(x_1 + x_2) = M_2(x_2) \geq 1$, $M_2(x_1 + x_2) = M_2(x_2) \geq M_2(y_2)$, and $M_2(y_1 + y_2) = \max(M_2(y_1), M_2(y_2))$, so $M_2(x_1+x_2) \geq M_2(y_1+y_2), M_2(x_1+x_2) \geq 1$. Thus $(x_1+x_2, y_1+y_2) \in S_2$.

Now note $\max(M_2(y_1), M_2(y_3)) = M_2(y_3)$ as $M_2(y_3) \geq 1 > M_2(y_1)$. Then $M_2(y_1 + y_3) = \max(M_2(y_1), M_2(y_3)) = M_2(y_3) \geq 1$, $M_2(y_1 + y_3) = M_2(y_3) > M_2(x_3)$ and $M_2(y_3) \geq 1 > M_2(x_1)$. Hence $M_2(y_1+y_3) > \max(M_2(x_1), M_2(x_3)) = M_2(x_1+x_3)$. Hence $(x_1+x_3, y_1+y_3) \in S_3$ as desired.

Solution 12. The area of ΔABC is equal to the area of the triangle with vertices $(0, 0)$, $B - A$, and $C - A$. By Problem 10, $B - A \in S_2$ and $C - A \in S_3$. Suppose $B - A = (x_2, y_2)$ and $C - A = (x_3, y_3)$. Then by the lemma, the area of the triangle is $|\frac{x_2 y_3 - x_3 y_2}{2}|$. We know $M_2(x_2) \geq M_2(y_2)$ and $M_2(y_3) > M_2(x_3)$, so $M_2(x_2 y_3) > M_2(x_3 y_2)$. By Problem 7, $M_2(\Delta ABC) = M_2(\frac{1}{2}) M_2(x_2 y_3 - x_3 y_2) = M_2(\frac{1}{2}) M_2(x_2 y_3) = 2 M_2(x_2) M_2(y_3) \geq 2 > 1$.

Solution 13. Without loss of generality, let the vertices be $A = (0, 0), B = (1, 0), C = (1, 1), D = (0, 1)$. Consider a triangulation T of $ABCD$, and place the vertices of T in to S_1, S_2, S_3. We observe that on edge AB we have $M_2(y) = 0$, on the edge BC we have $M_2(x) = 1$, on the edge CD we have $M_2(y) = 1$, and on the edge between DA we have $M_2(x) = 0$. Therefore edges between points in S_1 and points in S_2 on the boundary of $ABCD$ can only occur on edge AB (on edges BC and DA no points are in S_2 and on CD no points are in S_1). Furthermore, no point in AB is in S_3. Since $A \in S_1$ and $B \in S_2$ there are an odd number of such edges. Hence by Sperner's lemma, the number of complete triangles with respect to S_1, S_2, S_3 is odd and hence such a complete triangle exists. By problem 12, this triangle has area K with $M_2(K) > 1$. However, since $ABCD$ was split in m triangles of equal area we have $mK = 1$. Hence $M_2(mK) = M_2(m)M_2(K) = 1$ which implies $M_2(m) < 1$ and thus m is even.