CHMMC 2020-2021

Team Round

1. A unit circle is centered at $(0,0)$ on the (x,y) plane. A regular hexagon passing through $(1,0)$ is inscribed in the circle. Two points are randomly selected from the interior of the circle and horizontal lines are drawn through them, dividing the hexagon into at most three pieces. The probability that each piece contains exactly two of the hexagon's original vertices can be written as

$$\frac{2\left(\frac{m\pi}{n}+\frac{\sqrt{p}}{q}\right)^2}{\pi^2}$$

for positive integers m, n, p, and q such that m and n are relatively prime and p is squarefree. Find $m+n+p+q$.

2. Find the smallest positive integer k such that there is exactly one prime number of the form $kx+60$ for the integers $0 \leq x \leq 10$.

3. For any nonnegative integer n, let $S(n)$ be the sum of the digits of n. Let K be the number of nonnegative integers $n \leq 10^{10}$ that satisfy the equation

$$S(n) = (S(S(n)))^2.$$

Find the remainder when K is divided by 1000.

4. Select a random real number m from the interval $(\frac{1}{6}, 1)$. A track is in the shape of an equilateral triangle of side length 50 feet. Ch, Hm, and Mc are all initially standing at one of the vertices of the track. At the time $t=0$, the three people simultaneously begin walking around the track in clockwise direction. Ch, Hm, and Mc walk at constant rates of $2, 3$, and 4 feet per second, respectively. Let T be the set of all positive real numbers t_0 satisfying the following criterion:

If we choose a random number t_1 from the interval $[0, t_0]$, the probability that the three people are on the same side of the track at the time $t = t_1$ is precisely m.

The probability that $|T| = 17$ (i.e., T has precisely 17 elements) equals $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

5. Thanos establishes 5 settlements on a remote planet, randomly choosing one of them to stay in, and then he randomly builds a system of roads between these settlements such that each settlement has exactly one outgoing (unidirectional) road to another settlement. Afterwards, the Avengers randomly choose one of the 5 settlements to teleport to. Then, they (the Avengers) must use the system of roads to travel from one settlement to another. The probability that the Avengers can find Thanos can be written as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m+n$.

6. Suppose that

$$\prod_{n=1}^{\infty}\left(\frac{1+i\cot\left(\frac{n\pi}{2n+1}\right)}{1-i\cot\left(\frac{n\pi}{2n+1}\right)}\right)^{\frac{1}{n}} = \left(\frac{p}{q}\right)^{i\pi},$$

where p and q are relatively prime positive integers. Find $p+q$.

Note: for a complex number $z = re^{i\theta}$ for reals $r > 0, 0 \leq \theta < 2\pi$, we define $z^n = r^n e^{i\theta n}$ for all positive reals n.

7. For any positive integer n, let $f(n)$ denote the sum of the positive integers $k \leq n$ such that k and n are relatively prime. Let S be the sum of $\frac{1}{f(m)}$ over all positive integers m that are divisible by at least one of 2, 3, or 5, and whose prime factors are only 2, 3, or 5. Then $S = \frac{p}{q}$ for relatively prime positive integers p and q. Find $p + q$.

8. 15 ladies and 30 gentlemen attend a luxurious party. At the start of the party, each one of the ladies shakes hands with a random gentleman. At the end of the party, each of the ladies shakes hands with another random gentleman. A lady may shake hands with the same gentleman twice (first at the start and then at the end of the party), and no two ladies shake hands with the same gentleman at the same time.

Let m and n be relatively prime positive integers such that $\frac{m}{n}$ is the probability that the collection of ladies and gentlemen that shook hands at least once can be arranged in a single circle such that each lady is directly adjacent to someone if and only if she shook hands with that person. Find the remainder when m is divided by 10000.

9. Triangle ABC has circumcenter O and circumcircle ω. Let A_ω be the point diametrically opposite A on ω, and let H be the foot of the altitude from A onto BC. Let H_B and H_C be the reflections of H over B and C, respectively. Point P is the intersection of line $A_\omega B$ and the perpendicular of BC at point H_B, and point Q is the intersection of line $A_\omega C$ and the perpendicular of CB at point H_C. The circles ω_1 and ω_2 have the respective centers P and Q and respective radii PA and QA. Suppose that ω, ω_1, and ω_2 intersect at another common point X. If $AO = \frac{\sqrt{105}}{5}$ and $AX = 4$, then $|AB - CA|^2$ can be written as $m - n\sqrt{p}$ for positive integers m and n and squarefree positive integer p. Find $m + n + p$.

Note: the reflection of a point P over another point $Q \neq P$ is the point P' such that Q is the midpoint of P and P'.

10. Let ω be a nonreal 47th root of unity. Suppose that $\mathcal{S}$ is the set of polynomials of degree at most 46 and coefficients equal to either 0 or 1. Let N be the number of polynomials $Q \in \mathcal{S}$ such that

$$\sum_{j=0}^{46} \frac{Q(\omega^{2j}) - Q(\omega^j)}{\omega^{4j} + \omega^{3j} + \omega^{2j} + \omega^j + 1} = 47.$$

The prime factorization of N is $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$ where $p_1, \dots, p_s$ are distinct primes and $\alpha_1, \alpha_2, \dots, \alpha_s$ are positive integers. Compute $\sum_{j=1}^{s} p_j \alpha_j$.