CHMMC 2021-2022

Team Round Solutions

Problem 1. Let ABC be a right triangle with hypotenuse $\overline{AC}$ and circumcenter O. Point E lies on $\overline{AB}$ such that $AE = 9, EB = 3$; point F lies on $\overline{BC}$ such that $BF = 6, FC = 2$. Now suppose W, X, Y, and Z are the midpoints of $\overline{EB}$, $\overline{BF}$, $\overline{FO}$, and $\overline{OE}$, respectively. Compute the area of quadrilateral $WXYZ$.

Solution: $\boxed{12}$.
For completeness, we provide the proof of the following well-known fact:
Lemma 1: Consider a convex quadrilateral $ABCD$. Let X,Y,Z,W be the midpoints of AB,BC,CD,DA respectively. Then $[XYZW] = \frac{1}{2}[ABCD]$.
Proof: Put $A_1 = [ABD], A_2 = [CDB]$. By $\triangle ABD \sim \triangle AXW, \triangle CDB \sim \triangle CZY$ of ratio $\frac{1}{2}$, we have $[AXW] = \frac{1}{4}A_1, [CZY] = \frac{1}{4}A_2$, so $[AXW] + [CZY] = \frac{1}{4}[ABCD]$. Likewise, $[BYX] + [DWZ] = \frac{1}{4}[ABCD]$. Hence $[XYZW] = \frac{1}{2}[ABCD]$.
Lemma 1 implies that we just want to find

$$[XYZW] = \frac{1}{2}[EBFO] = \frac{1}{2}([OEB] + [OFB]) = \frac{1}{2}\left(\frac{1}{2}\cdot 4\cdot 3 + \frac{1}{2}\cdot 6\cdot 6\right) = \boxed{12}.$$

Problem 2. A prefrosh is participating in Caltech's "Rotation." They must rank Caltech's 8 houses, which are Avery, Page, Lloyd, Venerable, Ricketts, Blacker, Dabney, and Fleming, each a distinct integer rating from 1 to 8 inclusive. The conditions are that the rating x they give to Fleming is at most the average rating y given to Ricketts, Blacker, and Dabney, which is in turn at most the average rating z given to Avery, Page, Lloyd, and Venerable. Moreover x,y,z are all integers. How many such rankings can the prefrosh provide?

Solution: $\boxed{1296}$.
Remark

$$3y + 4z = 1 + \cdots + 8 - x = 36 - x$$

If $x \geq 5$ then $31 \geq 36 - x \geq (3+4)5 = 35$, contradiction. So $1 \leq x \leq 4$. Thus, one checks that the solutions for (x,y,z) are $(1,5,5), (3,3,6), (4,4,5)$.
For $(x,y,z) = (1,5,5)$: we have that this holds iff the numbers $\{a,b,c\}$ assigned to the houses corresponding to y among $\{1,\ldots,8\} \setminus \{x\}$ sum to 15. We can see that there are 5 such ways:

$$\{a,b,c\} = \{2,5,8\}, \{2,6,7\}, \{3,4,8\}, \{3,5,7\}, \{4,5,6\}.$$

For $(x,y,z) = (3,3,6)$: we can find 1 such way: $\{a,b,c\} = \{1,2,6\}$.
For $(x,y,z) = (4,4,5)$: we can find 3 such ways: $\{a,b,c\} = \{1,3,8\}, \{1,5,6\}, \{2,3,7\}$.
Given a proper ranking of houses, there are 3! ways to permute the numbers assigned to the houses corresponding to y, and there are 4! ways to permute the number assigned to the houses corresponding to z. These permutations leave y,z unchanged. So in total there are $(5+1+3)\cdot 3!\cdot 4! = \boxed{1296}$ such proper rankings.

Problem 3. Suppose a,b,c are complex numbers with $a+b+c = 0, a^2+b^2+c^2 = 0$, and $|a|,|b|,|c| \leq 5$. Suppose further at least one of a,b,c have real and imaginary parts that are both integers. Find the number of possibilities for such ordered triples (a,b,c).

Solution: $\boxed{481}$.
Note $2(ab+bc+ca) = (a+b+c)^2 - a^2 + b^2 + c^2 = 0$. Thus, by Vieta's formulas, a,b,c are the roots of $p(t) = t^3 - k$ for some complex constant k. Hence, if a is a root of $p(t)$, then $\omega a, \omega^2 a$ are necessarily the other two roots of $p(t)$, where $\omega = e^{2\pi i/3}$ denotes a third root of unity.

If one of a,b,c is 0, this forces $a=b=c=0$, yielding 1 solution.

Otherwise, take $z\in\mathbb{Z}[i],|z|\le 5$ ($\mathbb{Z}[i]$ denotes the subset of complex numbers with integral real and imaginary parts). The multiplication by ω map acts as rotation by 120° about the complex origin. Noticing $\tan(120^\circ)=-\sqrt{3}$ is irrational, the following lemma implies $\omega z,\omega^2 z\notin\mathbb{Z}[i]$.

Lemma 1: Let A,B be plane lattice points not at the origin O. Then, $\tan(\angle AOB)$ is rational.

Proof 1: The angle determined by $\overline{AO}$ and the x-axis has rational tan, where we take infinity to be rational. The same statement is true with the angle determined by $\overline{BO}$ and the x-axis. Thus, by the tangent angle addition formula, $\tan(\angle AOB)$ is rational.

It follows (a,b,c) is necessarily a permutation of $(z,\omega z,\omega^2 z)$. There are 81 plane lattice points inside or on the border of the circle centered at the origin of radius 5. This yields 80 choices of z, and thus $80\cdot 3!=480$ choices of (a,b,c) in this case.

Hence, the answer is $480+1=\boxed{481}$.

Problem 4. How many ordered triples (a,b,c) of integers $1\le a,b,c\le 31$ are there such that the remainder of $ab+bc+ca$ divided by 31 equals 8?

Solution: $\boxed{930}$.

The above condition is equivalent to $(a+c)(b+c)\equiv c^2+8 \pmod{31}$. Note $15^2\equiv 8 \pmod{31}$, so $\left(\frac{8}{31}\right)=1$. Since 31 is a prime $\equiv 3 \pmod 4$, we have $\left(\frac{-8}{31}\right)=-1$. Thus, $c^2+8\equiv 0 \pmod{31}$ never holds. Moreover, the equation $xy\equiv k \pmod p$ has $p-1$ solutions in integers $(x,y) \pmod p$ for any prime p and $k\not\equiv 0 \pmod p$.

Thus, for every residue $c \pmod{31}$, there are 30 solutions $(a+c,b+c)$ (and hence (a,b)) to $(a+c)(b+c)\equiv c^2+8 \pmod{31}$. Therefore, the answer is $31\cdot 30=\boxed{930}$.

Problem 5. How many cubics in the form $x^3-ax^2+(a+d)x-(a+2d)$ for integers a,d have roots that are all non-negative integers?

Solution: $\boxed{5}$.

Let the roots be r,s,t which are all non-negative integers. Then by Vieta's, $r+s+t=a, rs+st+tr=a+d, rst=a+2d \iff rst+r+s+t=2(rs+st+tr)$. Consider the following two cases:

Case 1: One of r,s,t is 0: WLOG say $r=0$. Then, it is necessary that $2st=s+t$, or equivalently $(2s-1)(2t-1)=1$. This means $s=t=1$ or $s=t=0$, yielding 2 different cubics.

Case 2: All of r,s,t are nonzero: Hence, $r,s,t\ge 1$. WLOG $r\ge s\ge t$. The above gives

$$rst<2(rs+st+tr) \implies \frac{3}{t}\ge\frac{1}{r}+\frac{1}{s}+\frac{1}{t}>\frac{1}{2}$$

Also, since $r+s+t\le rs+st+tr$, we have

$$2(rs+st+tr)\le rs+st+tr+rst \implies \frac{1}{r}+\frac{1}{s}+\frac{1}{t}\le 1$$

Hence, $t=2,3,4,5$. We enumerate the cases:

(1) $t=5$: Then $5rs+r+s+5=2(rs+5r+5s)$, so $3rs-9r-9s+5=0$, which is not possible $\pmod 3$.

(2) $t=4$: Then, $4rs+r+s+4=2(rs+4r+4s) \implies 4rs-14r-14s+8=0 \implies (2r-7)(2s-7)=41$. Since $r\ge s$, and (we can check) it is not possible for the factorization to be negative, we have $2r-7=41, 2s-7=1$, so $(r,s)=(24,4)$.

(3) $t=3$: Then $3rs+r+s+3=2(rs+3r+3s) \implies rs-5r-5s=-3 \implies (r-5)(s-5)=22$. Since $r\ge s$, and (we can check) it is not possible for the factorization to be negative, we have $r-5=22, s-5=1$, whence $(r,s)=(27,6)$, or $r-5=11, s-5=2$, whence $(r,s)=(16,7)$.

(4) $t=2$: Then $2rs+r+s+2=2(rs+2r+2s) \implies 3r+3s=2$, contradiction as $r,s\ge t>1$.

So this case gives 3 solutions.

In total, the answer is $\boxed{5}$, as all our solutions are distinct.

Problem 6. There is a unique degree-10 monic polynomial with integer coefficients $f(x)$ such that

$$f\left(\sum_{j=0}^{9} \sqrt[10]{2021^j}\right) = 0.$$

Find the remainder when $f(1)$ is divided by 1000.

Solution: $\boxed{91}$.

Put $\theta = \sqrt[10]{2021}, \alpha = \sum_{j=0}^{9} \sqrt[10]{2021^j} = \sum_{j=0}^{9} \theta^j$. Observe by geometric series $\alpha(\theta-1) = 2021-1 = 2020$, so $\alpha = \frac{2020}{\theta-1}$.

Note that θ is a root of the polynomial $g_1(x) = x^{10} - 2021$. We may transform the polynomial g_1 as follows. First, $\theta - 1$ is a root of

$$g_2(x) = g_1(x+1) = (x+1)^{10} - 2021 = -2020 + \sum_{j=1}^{10} \binom{10}{j} x^j.$$

Then, $\frac{1}{\theta-1}$ is a root of

$$g_3(x) = -x^{10} g_2\left(\frac{1}{x}\right) = 2020x^{10} - \sum_{j=1}^{10} \binom{10}{j} x^{10-j}$$

Finally, $\alpha = \frac{2020}{\theta-1}$ is a root of the monic

$$f(x) = 2020^9 g_3\left(\frac{x}{2020}\right) = x^{10} - \sum_{j=1}^{10} \binom{10}{j} 2020^{j-1} x^{10-j} = \frac{2020x^{10} + 1 - \sum_{j=0}^{10} \binom{10}{j} 2020^j x^{10-j}}{2020}.$$

Noting the fact $\sum_{j=0}^{10} \binom{10}{j} 2020^j x^{10-j} = (2020+x)^{10}$, we deduce

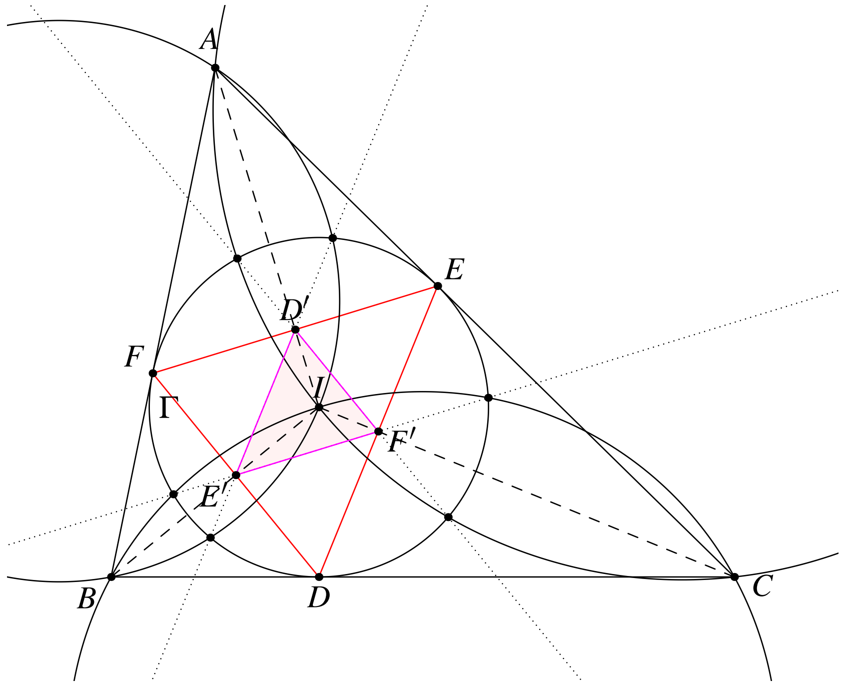
$$f(1) = \frac{2021 - 2021^{10}}{2020} = -2021(1 + 2021 + 2021^2)(1 + 2021^3 + 2021^6)$$
$$\implies f(1) \equiv \boxed{91} \pmod{1000}.$$

Remark: $f(x)$ is necessarily unique, as it is in fact the *minimal polynomial* of α over $\mathbb{Q}[x]$.

Problem 7. Let ABC be a triangle with $AB = 5$, $BC = 6$, and $CA = 7$. Denote Γ the incircle of ABC; let I be the center of Γ. The circumcircle of BIC intersects Γ at X_1 and X_2. The circumcircle of CIA intersects Γ at Y_1 and Y_2. The circumcircle of AIB intersects Γ at Z_1 and Z_2. The area of the triangle determined by $\overline{X_1X_2}$, $\overline{Y_1Y_2}$, and $\overline{Z_1Z_2}$ equals $\frac{m\sqrt{p}}{n}$ for positive integers m, n, and p, where m and n are relatively prime and p is squarefree. Compute $m+n+p$.

Solution: $\boxed{53}$.

Let D, E, and F be the points at which Γ touches BC, CA, and AB respectively. Denote $D' = \overline{AI} \cap \overline{EF}$, $E' = \overline{BI} \cap \overline{FD}$, and $F' = \overline{CI} \cap \overline{DE}$, as shown below.



Suppose $X = \overline{Y_1Y_2} \cap \overline{Z_1Z_2}$ and similarly define Y, Z. Note $D'E \cdot D'F = D'I \cdot D'A$, as $AEIF$ is cyclic, which means D' is on the radical axis of Γ and (AIB) and also on the radical axis of Γ and (AIC). Hence D' lies on $\overline{Y_1Y_2}$ and $\overline{Z_1Z_2}$. This means $D' = X$, so $X = \overline{AI} \cap \overline{EF}$, which is the midpoint of $\overline{EF}$. We have similar results for Y, Z. Since $\triangle D'E'F'$ is the medial triangle of $\triangle DEF$, the requested answer is $\frac{1}{4}[DEF]$.

By Heron's formula, $[ABC] = 6\sqrt{6}$. Furthermore, $AE = AF = 3$, $BF = BD = 2$, and $CD = CE = 4$. Hence, $[AEF] = \frac{3}{7} \cdot \frac{3}{5} \cdot [ABC]$; similar formulas apply for $[BFD]$ and $[CDE]$. Therefore,

$$\begin{aligned}[DEF] &= [ABC] - [AEF] - [BFD] - [CDE] \\ &= [ABC]\left(1 - \frac{3}{7} \cdot \frac{3}{5} - \frac{2}{5} \cdot \frac{2}{6} - \frac{4}{6} \cdot \frac{4}{7}\right) = \frac{48\sqrt{6}}{35}.\end{aligned}$$

Thus, $\frac{1}{4}[DEF] = \frac{12\sqrt{6}}{35}$. The answer is $\boxed{53}$.

Remark: the points X, Y, Z as above may alternatively be characterized via inversion $\mathscr{I}$ around Γ. Under $\mathscr{I}$, points $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ are held fixed, so $\mathscr{I} : (BIC) \mapsto \overline{X_1X_2}, (CIA) \mapsto \overline{Y_1Y_2}, (AIB) \mapsto \overline{Z_1Z_2}$. Moreover, $\mathscr{I} : A \mapsto D', B \mapsto E', C \mapsto F'$, so in fact $\overline{E'F'} = \overline{X_1X_2}$, $\overline{F'D'} = \overline{Y_1Y_2}$, and $\overline{D'E'} = \overline{Z_1Z_2}$. Thus, the triangle described by lines $\overline{X_1X_2}$, $\overline{Y_1Y_2}$, and $\overline{Z_1Z_2}$ is $\triangle D'E'F'$.

Problem 8. Depei is imprisoned by an evil wizard and is coerced to play the following game. Every turn, Depei flips a fair coin. Then, the following events occur in this order:

- The wizard computes the difference between the total number of heads and the total number of tails Depei has flipped. If that number is greater than or equal to 4 or less than or equal to -3, then Depei is vaporized by the wizard.
- The wizard determines if Depei has flipped at least 10 heads or at least 10 tails. If so, then the wizard releases Depei from the prison.

The probability that Depei is released by the evil wizard equals $\frac{m}{2^k}$, where m, k are positive integers. Compute $m + k$.

Solution: $\boxed{27184}$.

After 16 consecutive coin flips, Depei wins the game in (and only in) the following three conditions:

- He reaches 9 heads and 7 tails.
- He reaches 8 heads and 8 tails.
- He reaches 7 heads and 9 tails and immediately flips heads thereafter.

In what follows, encode the sequence of coin flips by a *north-east lattice paths* on $\mathbb{Z}^2$ starting from $(0,0)$, where a flip of heads indicates a unit east and a flip of tails indicates a unit north. We call any north-east lattice path $(0,0) \to (x_1,y_1)$ that does not touch the lines $y = x-4, y = x+3$ *proper*. Denote by $S_{(x_1,y_1)}$ be the number of proper lattice paths $(0,0) \to (x_1,y_1)$. To this end, we need to calculate $S_{(9,7)}, S_{(8,8)}, S_{(7,9)}$. We will use complementary counting.

Any lattice path $(0,0) \to (9,7)$ touching $y = x-4$ has a unique first touch-point. By reflecting the part of the lattice path after this first touch-point, one obtains a lattice path $(0,0) \to (11,5)$. Indeed, the collection of lattice paths $(0,0) \to (9,7)$ touching $y = x-4$ bijects to the collection of lattice paths $(0,0) \to (11,5)$ via this reflection. There are $\binom{16}{5}$ such paths. Similarly, a lattice path $(0,0) \to (9,7)$ touching $y = x+3$ has a unique first touch-point, and reflecting the part of the lattice path after the first touch-point yields a lattice path $(0,0) \to (4,12)$. The collection of lattice paths $(0,0) \to (9,8)$ touching $y = x+3$ bijects to the collection of lattice paths $(0,0) \to (4,12)$, and there are $\binom{16}{4}$ such paths.

We must also count the number of lattice paths $(0,0) \to (9,7)$ that touch both lines $y = x-4, y = x+3$. Order the touch-points of such a lattice path with the two lines in the obvious way; notice that its touch-points with the line $y = x-4$ must either precede or succeed *all* its touch-points with the line $y = x+3$. For instance, the lattice path cannot touch $y = x-4$, touch $y = x+3$, then touch $y = x-4$.

To this end, consider the collection of lattice paths $(0,0) \to (9,7)$ that has some touch-point of $y = x-4$ after some touch-point of $y = x+3$. The line $y = x-4$ reflected over $y = x+3$ is the line $y = x+10$. Thus, the above collection of lattice paths are in one-to-one correspondence with the collection of lattice paths $(0,0) \to (4,12)$ touching $y = x+10$, which in turn biject, via reflection, to the collection of lattice paths $(0,0) \to (2,14)$. There are $\binom{16}{2}$ such paths. By a similar argument, there is clearly 1 lattice path $(0,0) \to (9,7)$ that has some touch-point of $y = x+3$ after some touch-point of $y = x-4$. This covers all cases of lattice paths $(0,0) \to (9,7)$ touching at least one of the two lines.

We may apply a similar argument to count the lattice paths $(0,0) \to (8,8)$ and $(0,0) \to (7,9)$ touching at least one of the two lines. By complementary counting and PIE we have

$$S_{(9,7)} = \binom{16}{7} - \binom{16}{5} - \binom{16}{4} + \binom{16}{2} + 1$$
$$S_{(8,8)} = \binom{16}{8} - \binom{16}{5} - \binom{16}{4} + \binom{16}{1} + \binom{16}{1}$$
$$S_{(7,9)} = \binom{16}{7} - \binom{16}{6} - \binom{16}{3} + \binom{16}{2} + 1$$

and the probability of Depei winning is therefore

$$\frac{1}{2^{16}} \cdot \left(S_{(9,7)} + S_{(8,8)} + \frac{S_{(7,9)}}{2} \right) = \frac{27167}{2^{17}}$$

which gives an answer of $\boxed{27184}$.

Problem 9. Find the largest prime divisor of

$$\sum_{n=3}^{30} \binom{\binom{n}{3}}{2}.$$

Solution: $\boxed{431}$.

Note

$$E_n := \binom{\binom{n}{3}}{2}$$

equals the number of ways to pick two distinct 3-element subsets $A, B \subset \{1, \ldots, n\}$. The union $A \cup B$ describes either a 4-element, 5-element, or a 6-element subset of $\{1, \ldots, n\}$.

(1) For any 4-element set $\{a,b,c,d\}$, any of its 6 unordered pairs of 3-element subsets cover $\{a,b,c,d\}$.
(2) For any 3-element subset $A \subset \{a,b,c,d,e\}$, there are exactly three choices of another 3-element subset $B \subset \{a,b,c,d,e\}$ such that $A \cup B = \{a,b,c,d,e\}$. This counts $10 \cdot 3 = 30$ pairs of 3-element subsets of $\{a,b,c,d,e\}$, where each unordered pair is counted exactly twice. Thus, for any 5-element set $\{a,b,c,d,e\}$, there are 15 unordered pairs of 3-element subsets that cover $\{a,b,c,d,e\}$.
(3) For any 3-element subset $A \subset \{a,b,c,d,e,f\}$, there is exactly one choice of another 3-element subset $B \subset \{a,b,c,d,e,f\}$ such that $A \cup B = \{a,b,c,d,e,f\}$. Since there are $\binom{6}{3} = 20$ 3-element subsets of $\{a,b,c,d,e,f\}$, there are 10 unordered pairs of 3-element subsets of that cover $\{a,b,c,d,e,f\}$.

It follows that

$$E_n = 6\binom{n}{4} + 15\binom{n}{5} + 10\binom{n}{6}$$

and so by the hockey stick and Pascal identities

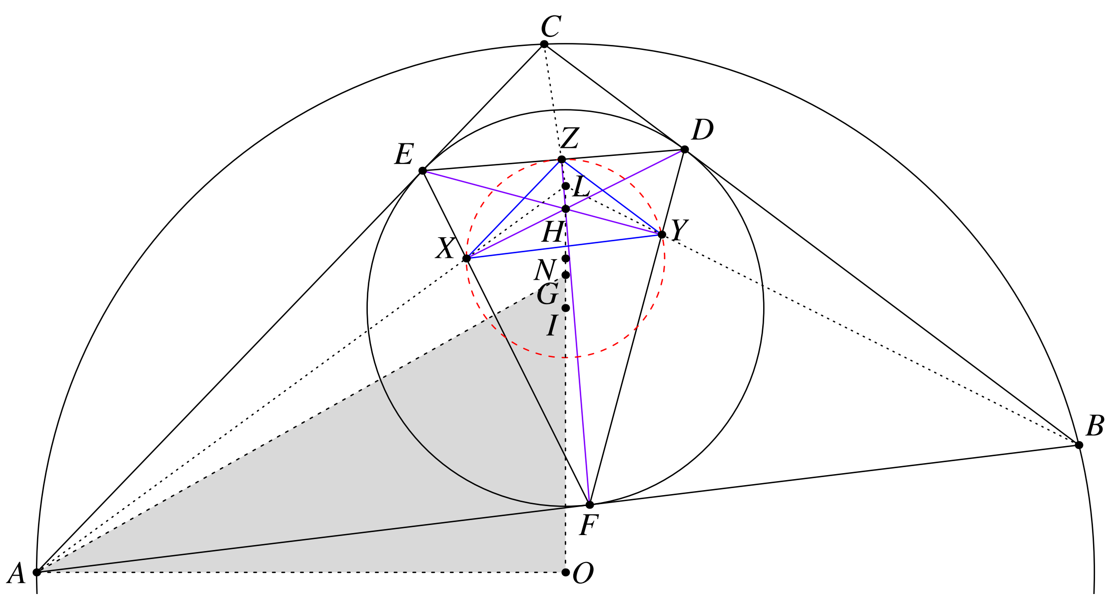
$$\sum_{n=3}^{30} E_n = \sum_{n=3}^{30} 6\binom{n}{4} + 15\binom{n}{5} + 10\binom{n}{6} = 6\binom{31}{5} + 15\binom{31}{6} + 10\binom{31}{7} = 6\binom{33}{7} + 3\binom{32}{7} + \binom{31}{7}.$$

With some arithmetic this simplifies readily to $3^2 \cdot 11 \cdot 29 \cdot 31 \cdot 431$, so the answer is $\boxed{431}$.

Problem 10. In triangle ABC, let O be the circumcenter. The incircle of ABC is tangent to $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at points D, E, and F, respectively. Let G be the centroid of triangle DEF. Suppose the inradius and circumradius of ABC is 3 and 8, respectively. Over all such triangles ABC, pick one that maximizes the area of triangle AGO. If we write $AG^2 = \frac{m}{n}$ for relatively prime positive integers m and n, then find m.

Solution: $\boxed{337}$.

We will show AG depends only on the inradius and circumradius of $\triangle ABC$. Let X, Y, and Z be the feet of the D-, E-, and F-altitudes of $\triangle DEF$ on $\overline{EF}$, $\overline{FD}$, and $\overline{DE}$, as shown below.



Note $\measuredangle FXY = \measuredangle EDF = \measuredangle EFA = \measuredangle XFA \implies \overline{XY} \parallel \overline{AB}$. Similarly, $\overline{YZ} \parallel \overline{BC}$ and $\overline{ZX} \parallel \overline{CA}$. Hence, $\triangle XYZ$ and $\triangle ABC$ are homothetic with center $L = \overline{AX} \cap \overline{BY} \cap \overline{CZ}$. Denote H and N the orthocenter and nine-point center of $\triangle DEF$, respectively. Note H and N is also the incenter and circumcenter of the orthic $\triangle XYZ$, respectively. Since I is the incenter of $\triangle ABC$, L, H, I are collinear by homothety. Likewise, O is the circumcenter of $\triangle ABC$, so L, N, O are collinear by homothety. Finally, I is also the circumcenter of $\triangle DEF$, so $\overline{IH}$ is the Euler line of $\triangle DEF$. Hence, I, H, N, G are collinear. Combining these three collinearities shows L, H, N, G, I, O are all collinear.

Note (XYZ) is the nine-point circle of $\triangle DEF$; hence, the radius of (XYZ) is $\frac{3}{2}$—half the inradius $r = 3$ of $\triangle ABC$, or the radius of (DEF). Since the circumradius of $\triangle ABC$ is $R = 8$, the ratio of homothety between $\triangle XYZ$ and $\triangle ABC$ is $\frac{\frac{3}{2}}{8} = \frac{3}{16}$. Assume $HI = LI - LH = 78x$ for some $x > 0$. With the well-known properties of the Euler line and the aforementioned homothety we compute

$$\begin{aligned} 3LI = 16LH &\implies LH = 18x,\ LI = 96x, \\ HN = NI = \frac{HI}{2} = 39x &\implies LN = 57x, \\ 3LO = 16LN &\implies LO = 304x \implies IO = LO - LI = 208x, \\ 3GI = HI &\implies GI = 26x \implies GO = IO + GI = 234x. \end{aligned}$$

Thus,

$$\frac{GO}{IO} = \frac{234x}{208x} = \frac{9}{8}.$$

By Euler's Theorem, $IO = \sqrt{R(R-2r)} = 4$. Thus, $GO = \frac{9}{8} \cdot IO = \frac{9}{2}$. To maximize the area of $\triangle AGO$, we must maximize the length of the A-altitude by "positioning" A on (ABC) such that $\overline{AO} \perp \overline{OG}$—such a $\triangle ABC$ clearly exists. Hence, $AG^2 = AO^2 + OG^2 = 64 + \frac{81}{4} = \frac{337}{4}$; the answer is $\boxed{337}$.