Power Round

CHMMC 2018

December 2nd, 2018

Welcome to the 2018 CHMMC Power Round! Have fun with these problems.

1 Euclidean Algorithm (25 pts)

Definition 1.1 (Greatest Common Divisor)**.** The greatest common divisor of two positive integers a and b, denoted $\gcd(a, b)$, is defined to be the greatest positive integer d such that $d \mid a$ and $d \mid b$.

Remark 1.2. The definition of divisibility is $d \mid a$ if and only if there exists an integer q such that $a = qd$.

/2 pts **Problem 1.1.** Prove that if a and b are positive integers such that $a > b$, then $\gcd(a, b) = \gcd(a - b, b)$.

/4 pts **Problem 1.2.** Prove that if a and b are positive integers such that $a = bq + r$ where $0 \le r < b$, then $\gcd(a, b) = \gcd(b, r)$.

Remark 1.3 (Division Algorithm)*.* For two positive integers a, b, there exists a *unique* quotient and remainder q and r such that $a = bq + r$ where $0 \le r < b$.

/3 pts **Problem 1.3** (Euclidean Algorithm)**.** To calculate the greatest common divisor of two positive integers a and b, we repeatedly apply the division algorithm to obtain a sequence of quotients $q_1, q_2, \ldots$ and remainders $r_1, r_2, \ldots$ such that

$$\begin{aligned} a &= bq_1 + r_1, \quad 0 \le r_1 < b \\ b &= r_1 q_2 + r_2, \quad 0 \le r_2 < r_1 \\ r_1 &= r_2 q_3 + r_3, \quad 0 \le r_3 < r_2 \end{aligned}$$

and so on, for $k \ge 3$,

$$r_{k-2} = r_{k-1} q_k + r_k, \quad 0 \le r_k < r_{k-1}.$$

Prove this process terminates after finitely many steps, at which point the remainder is zero, that is, $r_{n-1} = r_n q_{n+1}$ for some n. Prove that $r_n = \gcd(a, b)$.

/3 pts **Problem 1.4.** Compute $\gcd(100631, 423041)$ using the Euclidean Algorithm.

Definition 1.4 (Game of Euclid)**.** Two players A and B play the following game, where players alternate taking turns, with A going first. The game begins with two positive integers $a > b$. In a turn, a player replaces the larger number by subtracting from it a multiple of the smaller number, such that the result is nonnegative. Play continues until one of the numbers remaining is zero, then the last player to take a turn wins.

Remark 1.5. The description of this game is from a paper by Cole and Davie.

/3 pts **Problem 1.5.** Determine the player with the winning strategy in the Game of Euclid for $(a, b) = (162, 100)$ and $(a, b) = (161, 100)$.

Definition 1.6 (Golden Ratio)**.** The two roots of the quadratic $x^2 - x - 1 = 0$ are $\varphi = (1 + \sqrt{5})/2 \approx 1.618$ and $\psi = (1 - \sqrt{5})/2$.

/5 pts **Problem 1.6.** Prove that if $1 < a/b < \varphi$, there is only one possible move $(a, b) \to (b, a')$, and this satisfies $b/a' > \varphi$.

/5 pts **Problem 1.7.** Prove that player A may force a win if $a/b = 1$ or $a/b > \varphi$.

2 Fibonacci Numbers (38 pts)

Definition 2.1. The Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. For instance, $F_3 = 2, F_4 = 3, F_5 = 5$, and so forth.

/5 pts **Problem 2.1.** Let f_n be the number of ways to tile a board of size $n \times 1$ with squares (size one) and dominoes (size two). Prove $f_n = F_{n+1}$.

Figure 1: Some tilings of a board of size four

Remark 2.2. The tiling method is based on the book *Art of Combinatorial Proof* by Harvey Mudd professor Arthur Benjamin.

/9 pts **Problem 2.2.** Prove the following Fibonacci identities:

- $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$.
- $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$.
- $F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}$.

/6 pts **Problem 2.3.** Prove $F_{a+b} = F_{a+1}F_b + F_a F_{b-1}$ for $a, b \geq 1$.

Definition 2.3 (Fibonacci Nim)**.** Let there be n coins in a pile and A, B be two players who alternate removing coins from the pile, with A going first. On the first move, a player is not allowed to take all of the coins, and on each subsequent move, the number of coins removed can be at most twice that of the previous move. The winner is the player who removes the final coin(s).

/2 pt **Problem 2.4.** Demonstrate which player has the winning strategy in Fibonacci Nim for $n = 7, 10$.

/8 pts **Problem 2.5** (Zeckendorf's Theorem)**.** Prove that every positive integer N can be represented uniquely as a sum of distinct non-consecutive Fibonacci numbers F_k with $k \geq 2$.

Definition 2.4. Consider more general positions in Fibonacci Nim as pairs (q, r) consisting of a number of coins r remaining together with a "quota" q, specifying the maximum number of coins a player may take in the next move. Say that a position is *nice* if q is at least the smallest term in the Zeckendorf representation of r.

/8 pts **Problem 2.6.** (i) Show that given a nice position, there is a move such that the resulting position is not nice. (ii) Show that any move from a position which is not nice results in a nice position. (iii) Determine with proof the starting values of n for which the first player has no winning strategy.

3 Divisibility Sequences (27 pts)

Definition 3.1. A *divisibility sequence* is an integer sequence a_n for $n \geq 1$,

$$m \mid n \Rightarrow a_m \mid a_n.$$

/4 pts **Problem 3.1.** Prove the Fibonacci numbers are a divisibility sequence.

/3 pts **Problem 3.2.** Prove the sequence $a_n = A^n - B^n$ is divisibility for $A > B > 0$.

Definition 3.2. A divisibility sequence has *strong divisibility* if for all m, n positive integers, $\gcd(a_m, a_n) = a_{\gcd(m,n)}$.

/3 pts **Problem 3.3.** Prove the sequence $a_n = kn$ for natural k has strong divisibility.

/8 pts **Problem 3.4.** Prove $a_n = k^n - 1$ for natural k has strong divisibility. (*Hint:* There exist integers x and y such that $\gcd(m, n) = mx + ny$ by Bezout's.)

/9 pts **Problem 3.5.** Prove the Fibonacci Numbers have strong divisibility. (*Hint:* Show that if $n = qm + r$, then $\gcd(F_n, F_m) = \gcd(F_m, F_r)$.)