CHMMC 2023 Integration Bee Qualification Test Solutions

Problem 1. \( \int_{0}^{8} 5 \cdot x^3 \, dx \)

Proposed by Ritvik Teegavarapu

Solution: 96

This is a simple use of the power rule for integrals.

\[
\int_{0}^{8} 5 \cdot x^3 \, dx = \frac{5 \cdot x^4}{4} \bigg|_{0}^{8} = 3 \cdot 8^4 - 3 \cdot 0^4 = 3 \cdot 2^5 - 0 = 3 \cdot 32 = 96
\]

Problem 2. \( \int_{0}^{12} \frac{1}{(x-16) \ln 2} \, dx \)

Proposed by Ritvik Teegavarapu

Solution: -2

We can utilize regular integration properties as follows.

\[
\int_{0}^{12} \frac{1}{(x-16) \ln 2} \, dx = \ln \left| \frac{x-16}{\ln 2} \right|_{0}^{12}
\]

Evaluating, we have the following.

\[
\ln \left| \frac{12-16}{\ln 2} \right|_{0}^{12} = \ln \left| \frac{12-16}{\ln 2} \right| - \ln \left| \frac{0-16}{\ln 2} \right| = \ln \left( \frac{4}{\ln 2} \right) - \ln \left( \frac{16}{\ln 2} \right)
\]

We can use logarithm rules to prove the fact that \( \ln(2^a) = a \ln(2) \), and simplify as follows.

\[
\frac{\ln(2^2)}{\ln(2)} - \frac{\ln(2^4)}{\ln(2)} = 2 \frac{\ln(2)}{\ln(2)} - 4 \frac{\ln(2)}{\ln(2)} = 2 - 4 = -2
\]

Problem 3. \( \int_{-20}^{20} 20 - |x| \, dx \)

Proposed by Ritvik Teegavarapu

Solution: 400

This integral lends itself to a more geometric approach, in that the integrand can be decomposed into two triangles. Specifically, we have that one of the triangles is sloping upward with \( m = 1 \) on the interval to \([-20, 0]\), and one of the triangles is sloping downward with \( m = -1 \) on the interval to \([0, 20]\).

Therefore, we have two triangles, one on each sub-interval. Each of the triangles has a base of 20 since the sub-interval lengths are each 20, and height 20 since the \( y \)-intercept of the lines mentioned above is 20. Thus, adding the areas of both of these triangles, we have the following.

\[
\frac{400}{2} + \frac{400}{2} = 400
\]
The calculus-based approach is shown as follows.

\[
\int_{-20}^{20} 20 - |x| \, dx = \int_{-20}^{0} (20 + x) \, dx + \int_{0}^{20} (20 - x) \, dx
\]

\[
\int_{-20}^{0} (20 + x) \, dx = \left(20x + \frac{x^2}{2}\right) \bigg|_{-20}^{0} = 200
\]

\[
\int_{0}^{20} (20 - x) \, dx = \left(20x - \frac{x^2}{2}\right) \bigg|_{0}^{20} = 200
\]

\[
\int_{-20}^{20} 20 - |x| \, dx = 200 + 200 = 400
\]

**Problem 4.** \[ \int_{1}^{\infty} \frac{1}{\sqrt{x}(1+x)} \, dx \]

Proposed by Ritvik Teegavarapu

**Solution:** \( \frac{\pi}{2} \)

We seek to remove the \( \sqrt{t} \), so we split the fraction to set up the u-substitution.

\[
\int_{1}^{\infty} \frac{1}{\sqrt{t}(1+t)} \, dt = \int_{1}^{\infty} \frac{2}{2\sqrt{t}(1+t)} \, dt = \int_{1}^{\infty} \frac{1}{2\sqrt{t}} \cdot \frac{2}{1+t} \, dt
\]

Performing the u-substitution, we now have a much more recognizable integral.

\[
\int_{1}^{\infty} \frac{1}{2\sqrt{t}} \cdot \frac{2}{1+u^2} \, du = 2\arctan(u) \bigg|_{1}^{\infty}
\]

\[
2\arctan(u) \bigg|_{1}^{\infty} = 2\arctan(\infty) - 2\arctan(1) = 2 \cdot \left(\arctan(1)\right) - 2 \cdot \left(\arctan\left(\frac{1}{\sqrt{2}}\right)\right) = \frac{\pi}{2}
\]

**Problem 5.** \[ \int_{-\frac{1}{\sqrt{4 - x^2}}}^{1} \frac{1}{\sqrt{4 - x^2}} \, dx \]

Proposed by Brian Yang

**Solution:** \( \frac{\pi}{3} \)

With a \( u \)-substitution \( u = \frac{x}{2} \), which implies \( du = \frac{dx}{2} \), we have the following.

\[
\int_{-\frac{1}{\sqrt{4 - x^2}}}^{1} \frac{1}{\sqrt{4 - x^2}} \, dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{2}{\sqrt{4 - 4u^2}} \, du = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-u^2}} \, du
\]

The anti-derivative of the integrand is the \( \arcsin(x) \) function, so we compute as follows.

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-u^2}} \, du = \arcsin\left(\frac{1}{2}\right) - \arcsin\left(-\frac{1}{2}\right) = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}
\]
Problem 6.  \[
\int_0^2 (x^2 - 1) \cdot (x^3 - 3x)^{\frac{1}{2}} \, dx
\]

Proposed by Ritvik Teegavarapu

Solution: \[
\frac{2^\frac{7}{7}}{7}
\]

We recognize that this is a u-substitution, with \(u = x^3 - 3x\) and \(du = 3x^2 - 3 \, dx = 3(x^2 - 1) \, dx\). Thus, we have the following upon changing the bounds.

\[
\int_0^2 (x^2 - 1) \cdot (x^3 - 3x)^{\frac{1}{2}} \, dx \quad \overset{u=x^3-3x}{\longrightarrow} \quad \int_0^2 \frac{1}{3} \cdot (u)^{\frac{1}{2}} \, du
\]

This is a simple use of the power rule for integrals.

\[
\int_0^2 \frac{1}{3} \cdot (u)^{\frac{1}{2}} \, du = \frac{1}{3} \cdot \frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1} \bigg|_0^2 = \frac{1}{7}u^2 \bigg|_0^2 = \frac{2^\frac{7}{7}}{7}
\]

Problem 7.  \[
\int_1^2 \frac{2x^3 - 1}{x^4 + x} \, dx
\]

Proposed by Ritvik Teegavarapu

Solution: \(\ln(\frac{9}{4})\) or \(2 \cdot \ln(\frac{3}{2})\)

To induce a u-substitution, we consider dividing by \(x^2\) in the numerator and denominator, we have the following.

\[
\int_1^2 \frac{2x^3 - 1}{x^4 + x} \, dx = \int_1^2 \frac{2x - \frac{1}{x}}{x^2 + \frac{1}{x}}
\]

We now note that the numerator is simply the derivative of the denominator, as follows. Thus, we consider the \(u\)-substitution of the denominator \((x^2 + \frac{1}{x})\). This would change the bounds as follows.

\[
u = 1^2 + \frac{1}{1} = 2 \quad \quad u = 2^2 + \frac{1}{2} = \frac{9}{2}
\]

Thus, the new integral becomes the following.

\[
\int_1^2 \frac{2x - \frac{1}{x}}{x^2 + \frac{1}{x}} \, dx = \int_2^\frac{9}{2} \frac{du}{u} = \ln(u) \bigg|_2^\frac{9}{2} = \ln \left( \frac{9}{2} \right) - \ln(2) = \ln \left( \frac{9}{2} \right) - \ln \left( \frac{3^2}{2^2} \right) = 2 \cdot \ln \left( \frac{3}{2} \right)
\]

Problem 8.  \[
\int_0^{2024} x - \lfloor x \rfloor \, dx
\]

Proposed by Ritvik Teegavarapu

Solution: \(1012\)
We note that on the interval \([k, k + 1)\), we observe that \(\{x\}\) behaves like \(y = x\). Thus, we split the integral into intervals of length 1 as follows. Note that \(x - k\) is the line that is equivalent to \(\{x\}\) on the designated interval.

\[
\int_0^{2024} \{x\} \, dx = \sum_{k=0}^{2023} \int_k^{k+1} (x - k) \, dx
\]

Evaluating the integral, we have the following.

\[
\int_k^{k+1} (x - k) \, dx \quad \rightarrow \quad \int_0^1 x \, dx = \left. \frac{x^2}{2} \right|_0^1 = \frac{1}{2} - \frac{0^2}{2} = \frac{1}{2}
\]

Substituting this into our summation, we have the following.

\[
\sum_{k=0}^{2023} \int_k^{k+1} (x - k) \, dx = \sum_{k=0}^{2023} \frac{1}{2} (2023 - 0 + 1) = \frac{2024}{2} = 1012
\]

**Problem 9.** \(\int_0^\pi \frac{e^{\cos(x)}}{e^{\cos(x)} + e^{-\cos(x)}} \, dx\)

*Proposed by Ritvik Teegavarapu*

**Solution:** \(\frac{\pi}{2}\)

We can utilize King’s Rule, which is written below.

\[
\int_a^b f(x) \, dx = \int_a^b f(a + b - x) \, dx
\]

Thus, we label the initial integral as \(I\) and use King’s Rule to get an alternate integral as follows. We also note that \(\cos(\pi - x) = -\cos(x)\).

\[
I = \int_0^\pi \frac{e^{\cos(x)}}{e^{\cos(x)} + e^{-\cos(x)}} \, dx = \int_0^\pi \frac{e^{-\cos(x)}}{e^{-\cos(x)} + e^{\cos(x)}} \, dx
\]

Adding the two alternate forms of \(I\), we have the following.

\[
2I = \int_0^\pi \frac{e^{\cos(x)}}{e^{\cos(x)} + e^{-\cos(x)}} \, dx + \int_0^\pi \frac{e^{-\cos(x)}}{e^{-\cos(x)} + e^{\cos(x)}} \, dx
\]

Combining these two integrals together since they have the same bound and denominator, we have the following.

\[
2I = \int_0^\pi \frac{e^{\cos(x)} + e^{-\cos(x)}}{e^{\cos(x)} + e^{-\cos(x)}} \, dx
\]

We note that the numerator and the denominator are the same, meaning we have the following.

\[
2I = \int_0^\pi \frac{e^{\cos(x)} + e^{-\cos(x)}}{e^{\cos(x)} + e^{-\cos(x)}} \, dx = \int_0^\pi 1 \, dx = \left. x \right|_0^\pi = \pi \implies I = \frac{\pi}{2}
\]
Problem 10.\[ \int_{0}^{1} \binom{22}{20} x^2 (1-x)^{20} \, dx \]

Proposed by Ritvik Teegavarapu

Solution: \[ \frac{1}{23} \]

We can utilize a u-substitution of the form \( u = 1-x \), which implies \( du = -dx \).

\[
\int_{0}^{1} \binom{22}{20} x^2 (1-x)^{20} \, dx \quad \overset{u=1-x}{\Longrightarrow} \quad \int_{1}^{0} \left( \frac{22}{20} \right) (1-u)^2 u^{20} \, du = \int_{0}^{1} \left( \frac{22}{20} \right) (1-u)^2 u^{20} \, du
\]

Expanding the inside of the integral, we have the following. We note that the following holds true.

\[
\left( \frac{22}{20} \right) = \frac{22!}{2! \cdot 20!} = \frac{22 \cdot 21}{2} = 11 \cdot 21 = 231
\]

Substituting, we have the following.

\[
\int_{0}^{1} 231 (1-u)^2 u^{20} \, du = 231 \int_{0}^{1} (1-2u+u^2) \cdot u^{20} \, du = 231 \int_{0}^{1} u^{20} - 2u^{21} + u^{22} \, du
\]

Evaluating this integral, we have the following.

\[
231 \int_{0}^{1} u^{20} - 2u^{21} + u^{22} \, du = 231 \left[ \frac{u^{21}}{21} - \frac{2u^{22}}{22} + \frac{u^{23}}{23} \right]_0^1 = 231 \left[ \frac{1}{21} - \frac{1}{11} + \frac{1}{23} \right]
\]

Simplifying, we have the following.

\[
231 \left[ \frac{1}{21} - \frac{1}{11} + \frac{1}{23} \right] = 11 - 21 + \frac{230 + 1}{23} = -10 + 10 + \frac{1}{23} = \frac{1}{23}
\]

An alternate, and significantly more elegant, solution is recognizing that the integrand is strikingly similar to the definition of the Beta function, which evaluates as follows.

\[
\beta(a, b) = \int_{0}^{1} x^{a-1} (1-x)^{b-1} \, dx = \frac{(a-1)! (b-1)!}{(a+b-1)!}
\]

In our question, \( a = 3 \) and \( b = 21 \), which we can substitute and multiply accordingly by the binomial coefficient.

\[
\left( \frac{22}{20} \right) \cdot \beta(3, 21) = \left( \frac{22}{20} \right) \cdot \int_{0}^{1} x^2 (1-x)^{20} = \frac{22!}{20! \cdot 2! \cdot 23!} \cdot \frac{2! \cdot 20!}{23!} = \frac{22!}{23!} = \frac{1}{23}
\]

Problem 11. \[ \int_{1}^{e} x^{\ln(x)-1} \cdot \ln(x) \, dx \]

Proposed by Jeck Lim

Solution: \[ \frac{e-1}{2} \]
We note that $du = 1/\sqrt{x} \, dx$, so we can consider this as a $u$-substitution of $u = \ln(x)$ as follows. This also implies that $e^u = x$, which we can substitute in.

$$
\int_1^e \frac{x \ln(x) \cdot \ln(x)}{x} \, dx \implies \int_0^1 (e^u)^u \, u \, du = \int_0^1 u \cdot e^{u^2} \, du
$$

We can consider the $v$-substitution, in which we allow $v = u^2$. This implies that $dv = 2u \, du$. Thus, we have the following.

$$
\int_0^1 u \cdot e^{u^2} \, du \implies \int_0^1 e^v \cdot \left( \frac{dv}{2} \right) = \left( \frac{e^v}{2} \right) \bigg|_0^1 = \frac{e^1}{2} - \frac{e^0}{2} = \frac{e - 1}{2}
$$

**Problem 12.** $\int_0^e \sin(\ln(x)) \, dx$

*Proposed by Jeck Lim*

**Solution:** $\frac{e^π}{2}$

We can consider $x = e^u$, which implies $dx = e^u \, du$ and the following equivalent integral.

$$
\int_0^e \sin(\ln(x)) \, dx \implies \int_\infty^0 \sin(u) \cdot e^u \, du
$$

We can use integration by parts on this, with $a = \sin(u)$ and $db = e^u \, du$, which gives the following.

$$
\int a \, db = ab - \int b \, da = (\sin(u) \cdot e^u) \bigg|_0^\pi - \int_\infty^0 \cos(u) \cdot e^u \, du
$$

Doing integration by parts on this integral again, with $b = \cos(u)$ and $da = e^u \, du$, we have the following.

$$
\int b \, da = ab - \int a \, db = (\cos(u) \cdot e^u) \bigg|_0^\pi + \int_\infty^0 \sin(u) \cdot e^u \, du
$$

Thus, we have the following, where $I$ is the value of the initial integral.

$$
I = (\sin(u) \cdot e^u) \bigg|_0^\pi - \int_\infty^0 \cos(u) \cdot e^u \, du = (\sin(u) \cdot e^u) \bigg|_0^\pi - (\cos(u) \cdot e^u) \bigg|_\infty^\pi + I
$$

$$
2I = (\sin(u) \cdot e^u) \bigg|_\infty^\pi - (\cos(u) \cdot e^u) \bigg|_\infty^\pi = [(\sin(\pi)e^\pi) - (\sin(-\infty)e^{-\infty})] - [(\cos(\pi)e^\pi) - (\cos(-\infty)e^{-\infty})]
$$

All of the terms with $e^{-\infty}$ will vanish to 0. Thus, we have the following.

$$
[(\sin(\pi)e^\pi) - (\sin(-\infty)e^{-\infty})] - [(\cos(\pi)e^\pi) - (\cos(-\infty)e^{-\infty})] = 0 \cdot e^\pi - ((-1) \cdot e^\pi) = e^\pi
$$

Thus, the value of the integral $I$ can be calculated as follows.

$$
2I = e^\pi \implies I = \frac{e^\pi}{2}
$$
Problem 13. \( \int_1^e \frac{\ln(x)}{(\ln(x)+1)^2} \, dx \)

Proposed by Jeck Lim

Solution: \( \frac{e}{2} - 1 \)

We consider the \( u \)-substitution of \( x = e^u \) in order to eliminate the \( \ln(x) \) present, which implies that \( dx = e^u \, du \). Thus, we have the following equivalent integral.

\[
\int_1^e \frac{\ln(x)}{(\ln(x)+1)^2} \, dx \implies \int_0^1 \frac{u}{(u+1)^2} \cdot (e^u \, du)
\]

We can use partial fraction decomposition on the fraction as follows.

\[
\frac{ue^u}{(u+1)^2} = \frac{Ae^u}{u+1} + \frac{Be^u}{(u+1)^2}
\]

Solving for \( A \) and \( B \), we have the following.

\[
ue^u = Ae^u(u+1) + Be^u
\]

Thus, \( A = 1 \) and \( B = -1 \). Therefore, we can split the integral as follows.

\[
\int_0^1 \frac{u}{(u+1)^2} \cdot (e^u \, du) = \int_0^1 \frac{e^u}{u+1} \, du - \int_0^1 \frac{e^u}{(u+1)^2} \, du
\]

Performing integration by parts on the first integral, we have the following as our selection for \( a \) and \( b \). We allow \( a = 1/(u+1) \) and \( db = e^u \, du \).

\[
\int_0^1 \frac{e^u}{u+1} \, du = \int a \, db = ab - \int b \, da = \frac{e^u}{u+1} \bigg|_0^1 + \int_0^1 \frac{e^u}{(u+1)^2} \, du
\]

Simplifying, we have the following.

\[
\frac{e^u}{u+1} \bigg|_0^1 + \int_0^1 \frac{e^u}{(u+1)^2} \, du = \left( \frac{e^1}{1+1} \right) - \left( \frac{e^0}{0+1} \right) + \int_0^1 \frac{e^u}{(u+1)^2} \, du = \frac{e}{2} - 1 + \int_0^1 \frac{e^u}{(u+1)^2} \, du
\]

Substituting this equivalent expression, we note that the two integrals cancel, and we are left with the final result.

\[ I = \int_{0}^{1} \frac{u}{(u+1)^{2}} \cdot (e^u \, du) = \left( \frac{e}{2} - 1 + \int_{0}^{1} \frac{e^u}{(u+1)^{2}} \, du \right) - \int_{0}^{1} \frac{e^u}{(u+1)^{2}} \, du = \frac{e}{2} - 1 \]

Problem 14. \( \int_0^1 \left[ \sin \left( \frac{\pi}{x} \right) \right] \, dx \)

Proposed by Jeck Lim

Solution: \( \ln(1/2) \) or \( -\ln(2) \)

If we pick any \( x \in [1/2, 1] \), we note the following occurs at the bounds.

\[
\sin \left( \frac{\pi}{2} \right) = \sin(2\pi) = 0 \quad \sin \left( \frac{\pi}{1} \right) = \sin(\pi) = 0
\]
We now need to note if the signed area will be positive or negative. Between the angles of $\pi$ and $2\pi$, $\sin(x)$ will be negative, namely less than 0. Due to the floor function, this rounds down and we have that it evaluates to $-1$. Thus, on this interval of $[0.5, 1]$, the function in question will have a rectangle with width 0.5, as well as a height of $-1$ (this implicates that the rectangle is subtended \textbf{underneath} the $x$-axis).

Repeating this process on the interval of $x \in [1/3, 1/2]$, we note that the following occurs at the bounds.

$$\sin \left( \frac{\pi}{3} \right) = \sin(3\pi) = 0 \quad \sin \left( \frac{\pi}{2} \right) = \sin(2\pi) = 0$$

We now need to note if the signed area will be positive or negative. Between the angles of $2\pi$ and $3\pi$, $\sin(x)$ will be positive, namely greater than 0. Due to the floor function, this rounds down and we have that it evaluates to 0. Thus, on this interval of $[0.5, 1]$, the function in question will have a rectangle with width 0.5 and a height of 0.

We can now see the general pattern. For any interval $[1/k, 1/(k+1)]$, we have that if $k$ is odd the function evaluates to 0. If we instead have that $k$ is even, we have that the function evaluates to $-1$. The function only gives the height of the rectangle, which means the width of the rectangle can be calculated from the length of the interval, which is $1/k - 1/(k+1)$. Thus, the area summation is as follows.

$$\int_0^1 \left[ \sin \left( \frac{\pi}{x} \right) \right] \, dx = A = \sum_{k=1}^{\infty} (-1) \cdot \left( \frac{1}{k} - \frac{1}{k+1} \right) = - \left( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \right)$$

We can recognize this as the alternating harmonic series, which converges as follows.

$$A = - \left( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \right) = -(\ln(2)) = \ln(1/2) = \ln(2^{-1}) = -\ln(2)$$