CHMMC 2023 Integration Bee Qualification Test Solutions

Problem 1. $\int_0^8 5\cdot x^{\frac{2}{3}}\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{96}$.

This is a simple use of the power rule for integrals.

$$\int_0^8 5\cdot x^{\frac{2}{3}}\ \mathrm{dx} = \left.\frac{5\cdot x^{\frac{2}{3}+1}}{\frac{2}{3}+1}\right|_0^8 = \left.3x^{\frac{5}{3}}\right|_0^8 = 3\cdot(8)^{\frac{5}{3}} - 3\cdot(0)^{\frac{5}{3}} = 3\cdot(2)^5 - 0 = 3\cdot 32 = \boxed{96}$$

Problem 2. $\int_0^{12} \frac{1}{(x-16)\ln 2}\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{-2}$.

We can utilize regular integration properties as follows.

$$\int_0^{12} \frac{1}{(x-16)\cdot\ln(2)}\ \mathrm{dx} = \left.\frac{\ln(|x-16|)}{\ln(2)}\right|_0^{12}$$

Evaluating, we have the following.

$$\left.\frac{\ln(|x-16|)}{\ln(2)}\right|_0^{12} = \frac{\ln(|12-16|)}{\ln(2)} - \frac{\ln(|0-16|)}{\ln(2)} = \frac{\ln(4)}{\ln(2)} - \frac{\ln(16)}{\ln(2)}$$

We can use logarithm rules to prove the fact that $\ln(2^a) = a\ln(2)$, and simplify as follows.

$$\frac{\ln(2^2)}{\ln(2)} - \frac{\ln(2^4)}{\ln(2)} = \frac{2\ln(2)}{\ln(2)} - \frac{4\ln(2)}{\ln(2)} = 2 - 4 = \boxed{-2}$$

Problem 3. $\int_{-20}^{20} 20 - |x|\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{400}$.

This integral lends itself to a more geometric approach, in that the integrand can be decomposed into two triangles. Specifically, we have that one of the triangles is sloping upward with $m = 1$ on the interval to $[-20, 0]$, and one of the triangles is sloping downward with $m = -1$ on the interval to $[0, 20]$.

Therefore, we have two triangles, one on each sub-interval. Each of the triangles has a base of 20 since the sub-interval lengths are each 20, and height 20 since the y-intercept of the lines mentioned above is 20. Thus, adding the areas of both of these triangles, we have the following.

$$\frac{400}{2} + \frac{400}{2} = \boxed{400}$$

The calculus-based approach is shown as follows.

$$\int_{-20}^{20} 20-|x| \text{ dx} = \int_{-20}^{0} (20+x) \text{ dx} + \int_{0}^{20} (20-x) \text{ dx}$$

$$\int_{-20}^{0} (20+x) \text{ dx} = \left(20x+\frac{x^2}{2}\right)\Bigg|_{-20}^{0} = 200$$

$$\int_{0}^{20} (20-x) \text{ dx} = \left(20x-\frac{x^2}{2}\right)\Bigg|_{0}^{20} = 200$$

$$\int_{-20}^{20} 20-|x| \text{ dx} = 200+200 = 400$$

Problem 4. $\displaystyle\int_{1}^{\infty} \frac{1}{\sqrt{x}(1+x)} \text{ dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\dfrac{\pi}{2}}$.

We seek to remove the $\sqrt{t}$, so we split the fraction to set up the u-substitution.

$$\int_{1}^{\infty} \frac{1}{\sqrt{t}(1+t)} \text{ dt} = \int_{1}^{\infty} \frac{2}{2\sqrt{t}(1+t)} \text{ dt} = \int_{1}^{\infty} \frac{1}{2\sqrt{t}} \cdot \frac{2}{1+t} \text{ dt}$$

Performing the u-substitution, we now have a much more recognizable integral.

$$\int_{1}^{\infty} \frac{1}{2\sqrt{t}} \cdot \frac{2}{1+(\sqrt{t})^2} \text{ dt} \underbrace{\Longrightarrow}_{u=\sqrt{t}} \int_{1}^{\infty} \frac{2}{1+u^2} \text{ du} = 2\arctan(u)\Bigg|_{1}^{\infty}$$

$$2\arctan(u)\Bigg|_{1}^{\infty} = 2\arctan(\infty) - 2\arctan(1) = 2\cdot\left(\frac{\pi}{2}\right) - 2\cdot\left(\frac{\pi}{4}\right) = \boxed{\frac{\pi}{2}}$$

Problem 5. $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{4-x^2}} \text{ dx}$

Proposed by Brian Yang

Solution: $\boxed{\dfrac{\pi}{3}}$.

With a u-substitution $u = \frac{x}{2}$, which implies $\text{d}u = \frac{\text{d}x}{2}$, we have the following.

$$\int_{-1}^{1} \frac{1}{\sqrt{4-x^2}} \text{ d}x = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{2}{\sqrt{4-4u^2}} \text{ d}u = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-u^2}} \text{ d}u$$

The anti-derivative of the integrand is the $\arcsin(x)$ function, so we compute as follows.

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-u^2}} \text{ d}u = \arcsin\left(\frac{1}{2}\right) - \arcsin\left(-\frac{1}{2}\right) = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \boxed{\frac{\pi}{3}}$$

Problem 6. $\int_0^2 (x^2-1)\cdot(x^3-3x)^{\frac{4}{3}}\text{ dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\frac{2^{\frac{7}{3}}}{7}}$.

We recognize that this is a u-substitution, with $u = x^3-3x$ and $\text{du} = 3x^2-3\text{ dx} = 3(x^2-1)\text{ dx}$. Thus, we have the following upon changing the bounds.

$$\int_0^2 (x^2-1)\cdot(x^3-3x)^{\frac{4}{3}}\text{ dx} \underbrace{\Longrightarrow}_{u=x^3-3x} \int_0^2 \frac{1}{3}\cdot(u)^{\frac{4}{3}}\text{ du}$$

This is a simple use of the power rule for integrals.

$$\int_0^2 \frac{1}{3}\cdot(u)^{\frac{4}{3}}\text{ du} = \frac{1}{3}\cdot\frac{u^{\frac{4}{3}+1}}{\frac{4}{3}+1}\Bigg|_0^2 = \frac{1}{7}u^{\frac{7}{3}}\Bigg|_0^2 = \boxed{\frac{2^{\frac{7}{3}}}{7}}$$

Problem 7. $\int_1^2 \frac{2x^3-1}{x^4+x}\text{ dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\ln(9/4)}$ **or** $\boxed{2\cdot\ln(3/2)}$

To induce a u-substitution, we consider dividing by x^2 in the numerator and denominator, we have the following.

$$\int_1^2 \frac{2x^3-1}{x^4+x}\text{ dx} = \int_1^2 \frac{2x-\frac{1}{x^2}}{x^2+\frac{1}{x}}$$

We now note that the numerator is simply the derivative of the denominator, as follows. Thus, we consider the u-substitution of the denominator (x^2+1/x). This would change the bounds as follows.

$$u = 1^2+\frac{1}{1} = 2 \qquad\qquad u = 2^2+\frac{1}{2} = \frac{9}{2}$$

Thus, the new integral becomes the following.

$$\int_1^2 \frac{2x-\frac{1}{x^2}}{x^2+\frac{1}{x}}\text{ dx} = \int_2^{\frac{9}{2}} \frac{\text{du}}{u} = \ln(u)\Bigg|_2^{\frac{9}{2}} = \ln\left(\frac{9}{2}\right)-\ln(2) = \boxed{\ln\left(\frac{9}{4}\right)} = \ln\left(\frac{3^2}{2^2}\right) = \boxed{2\cdot\ln\left(\frac{3}{2}\right)}$$

Problem 8. $\int_0^{2024} x-\lfloor x\rfloor\text{ dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{1012}$

We note that on the interval $[k, k+1)$, we observe that $\{x\}$ behaves like $y = x$. Thus, we split the integral into intervals of length 1 as follows. Note that $x - k$ is the line that is equivalent to $\{x\}$ on the designated interval.

$$\int_0^{2024} \{x\} \text{ dx} = \sum_{k=0}^{2023} \int_k^{k+1} (x-k) \text{ dx}$$

Evaluating the integral, we have the following.

$$\int_k^{k+1} (x-k) \text{ dx} \underbrace{\implies}_{u=x-k} \int_0^1 x \text{ dx} = \frac{x^2}{2}\bigg|_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$$

Substituting this into our summation, we have the following.

$$\sum_{k=0}^{2023} \int_k^{k+1} (x-k) \text{ dx} = \sum_{k=0}^{2023} \frac{1}{2} = \frac{1}{2} \cdot (2023 - 0 + 1) = \frac{2024}{2} = \boxed{1012}$$

Problem 9. $\displaystyle\int_0^{\pi} \frac{e^{\cos(x)}}{e^{\cos(x)} + e^{-\cos(x)}} \text{ dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\dfrac{\pi}{2}}$

We can utilize King's Rule, which is written below.

$$\int_a^b f(x) \text{ dx} = \int_a^b f(a+b-x) \text{ dx}$$

Thus, we label the initial integral as I and use King's Rule to get an alternate integral as follows. We also note that $\cos(\pi - x) = -\cos(x)$.

$$I = \int_0^{\pi} \frac{e^{\cos(\pi-x)}}{e^{\cos(\pi-x)} + e^{-\cos(\pi-x)}} \text{ dx} = \int_0^{\pi} \frac{e^{-\cos(x)}}{e^{-\cos(x)} + e^{\cos(x)}} \text{ dx}$$

Adding the two alternate forms of I, we have the following.

$$2I = \int_0^{\pi} \frac{e^{\cos(x)}}{e^{\cos(x)} + e^{-\cos(x)}} \text{ dx} + \int_0^{\pi} \frac{e^{-\cos(x)}}{e^{-\cos(x)} + e^{\cos(x)}} \text{ dx}$$

Combining these two integrals together since they have the same bound and denominator, we have the following.

$$2I = \int_0^{\pi} \frac{e^{\cos(x)} + e^{-\cos(x)}}{e^{\cos(x)} + e^{-\cos(x)}} \text{ dx}$$

We note that the numerator and the denominator are the same, meaning we have the following.

$$2I = \int_0^{\pi} \frac{e^{\cos(x)} + e^{-\cos(x)}}{e^{\cos(x)} + e^{-\cos(x)}} \text{ dx} = \int_0^{\pi} 1 \text{ dx} = x\bigg|_0^{\pi} = \pi \implies I = \boxed{\frac{\pi}{2}}$$

Problem 10. $\int_0^1 \binom{22}{20} x^2(1-x)^{20}\, \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\dfrac{1}{23}}$

We can utilize a u-substitution of the form $u = 1-x$, which implies $\mathrm{du} = -\mathrm{dx}$.

$$\int_0^1 \binom{22}{20} x^2(1-x)^{20}\, \mathrm{dx} \underbrace{\Longrightarrow}_{u=1-x} \int_1^0 -\binom{22}{20}(1-u)^2u^{20}\, \mathrm{du} = \int_0^1 \binom{22}{20}(1-u)^2u^{20}\, \mathrm{du}$$

Expanding the inside of the integral, we have the following. We note that the following holds true.

$$\binom{22}{20} = \frac{22!}{2!\cdot 20!} = \frac{22\cdot 21}{2} = 11\cdot 21 = 231$$

Substituting, we have the following.

$$\int_0^1 231(1-u)^2u^{20}\, \mathrm{du} = 231\int_0^1 (1-2u+u^2)\cdot u^{20}\, \mathrm{du} = 231\int_0^1 u^{20} - 2u^{21} + u^{22}\, \mathrm{du}$$

Evaluating this integral, we have the following.

$$231\int_0^1 u^{20} - 2u^{21} + u^{22}\, \mathrm{du} = 231\cdot\left[\frac{u^{21}}{21} - \frac{2u^{22}}{22} + \frac{u^{23}}{23}\right]_0^1 = 231\cdot\left[\frac{1}{21} - \frac{1}{11} + \frac{1}{23}\right]$$

Simplifying, we have the following.

$$231\cdot\left[\frac{1}{21} - \frac{1}{11} + \frac{1}{23}\right] = 11 - 21 + \frac{230+1}{23} = -10 + 10 + \frac{1}{23} = \boxed{\frac{1}{23}}$$

An alternate, and significantly more elegant, solution is recognizing that the integrand is strikingly similar to the definition of the Beta function, which evaluates as follows.

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, \mathrm{dx} = \frac{(a-1)!(b-1)!}{(a+b-1)!}$$

In our question, $a = 3$ and $b = 21$, which we can substitute and multiply accordingly by the binomial coefficient.

$$\binom{22}{20}\cdot\beta(3,21) = \binom{22}{20}\cdot\int_0^1 x^2(1-x)^{20} = \frac{22!}{20!\cdot 2!}\cdot\frac{2!\cdot 20!}{23!} = \frac{22!}{23!} = \frac{1}{23}$$

Problem 11. $\int_1^e x^{\ln(x)-1}\cdot\ln(x)\, \mathrm{dx}$

Proposed by Jeck Lim

Solution: $\boxed{\dfrac{e-1}{2}}$

We note that du $= 1/x$ dx, so we can consider this as a u-substitution of $u = \ln(x)$ as follows. This also implies that $e^u = x$, which we can substitute in.

$$\int_1^e \frac{x^{\ln(x)} \cdot \ln(x)}{x} \text{ dx} \implies \int_0^1 (e^u)^u \cdot u \text{ du} = \int_0^1 u \cdot e^{u^2} \text{ du}$$

We can consider the v-substitution, in which we allow $v = u^2$. This implies that dv $= 2u$ du. Thus, we have the following.

$$\int_0^1 u \cdot e^{u^2} \text{ du} \implies \int_0^1 e^v \cdot \left(\frac{\text{dv}}{2}\right) = \left(\frac{e^v}{2}\right)\bigg|_0^1 = \frac{e^1}{2} - \frac{e^0}{2} = \boxed{\frac{e-1}{2}}$$

Problem 12. $\displaystyle\int_0^{e^\pi} \sin(\ln(x)) \text{ dx}$

Proposed by Jeck Lim

Solution: $\boxed{\dfrac{e^\pi}{2}}$

We can consider $x = e^u$, which implies dx $= e^u$ du and the following equivalent integral.

$$\int_0^{e^\pi} \sin(\ln(x)) \text{ dx} \implies \int_{-\infty}^{\pi} \sin(u) \cdot e^u \text{ du}$$

We can use integration by parts on this, with $a = \sin(u)$ and db $= e^u$ du, which gives the following.

$$\int a \text{ db} = ab - \int b \text{ da} = (\sin(u) \cdot e^u)\bigg|_{-\infty}^{\pi} - \int_{-\infty}^{\pi} \cos(u) \cdot e^u \text{ du}$$

Doing integration by parts on this integral again, with $b = \cos(u)$ and da $= e^u$ du, we have the following.

$$\int b \text{ da} = ab - \int a \text{ db} = (\cos(u) \cdot e^u)\bigg|_{-\infty}^{\pi} + \int_{-\infty}^{\pi} \sin(u) \cdot e^u \text{ du}$$

Thus, we have the following, where I is the value of the initial integral.

$$I = (\sin(u) \cdot e^u)\bigg|_{-\infty}^{\pi} - \int_{-\infty}^{\pi} \cos(u) \cdot e^u \text{ du} = (\sin(u) \cdot e^u)\bigg|_{-\infty}^{\pi} - \left((\cos(u) \cdot e^u)\bigg|_{-\infty}^{\pi} + I\right)$$

$$2I = (\sin(u) \cdot e^u)\bigg|_{-\infty}^{\pi} - (\cos(u) \cdot e^u)\bigg|_{-\infty}^{\pi} = \left[(\sin(\pi)e^\pi) - \left(\sin(-\infty)e^{-\infty}\right)\right] - \left[(\cos(\pi)e^\pi) - \left(\cos(-\infty)e^{-\infty}\right)\right]$$

All of the terms with $e^{-\infty}$ will vanish to 0. Thus, we have the following.

$$\left[(\sin(\pi)e^\pi) - \left(\sin(-\infty)e^{-\infty}\right)\right] - \left[(\cos(\pi)e^\pi) - \left(\cos(-\infty)e^{-\infty}\right)\right] = 0 \cdot e^\pi - ((-1) \cdot e^\pi) = e^\pi$$

Thus, the value of the integral I can be calculated as follows.

$$2I = e^\pi \implies I = \boxed{\frac{e^\pi}{2}}$$

Problem 13. $\int_1^e \frac{\ln(x)}{(\ln(x)+1)^2}\,\mathrm{dx}$

Proposed by Jeck Lim

Solution: $\boxed{\frac{e}{2}-1}$

We consider the u-substitution of $x = e^u$ in order to eliminate the $\ln(x)$ present, which implies that $\mathrm{dx} = e^u\,\mathrm{du}$. Thus, we have the following equivalent integral.

$$\int_1^e \frac{\ln(x)}{(\ln(x)+1)^2}\,\mathrm{dx} \implies \int_0^1 \frac{u}{(u+1)^2}\cdot(e^u\,\mathrm{du})$$

We can use partial fraction decomposition on the fraction as follows.

$$\frac{ue^u}{(u+1)^2} = \frac{Ae^u}{u+1} + \frac{Be^u}{(u+1)^2}$$

Solving for A and B, we have the following.

$$ue^u = Ae^u(u+1) + Be^u$$

Thus, $A = 1$ and $B = -1$. Therefore, we can split the integral as follows.

$$\int_0^1 \frac{u}{(u+1)^2}\cdot(e^u\,\mathrm{du}) = \int_0^1 \frac{e^u}{u+1}\,\mathrm{du} - \int_0^1 \frac{e^u}{(u+1)^2}\,\mathrm{du}$$

Performing integration by parts on the first integral, we have the following as our selection for a and b. We allow $\mathrm{a} = 1/(u+1)$ and $\mathrm{db} = e^u\,\mathrm{du}$.

$$\int_0^1 \frac{e^u}{u+1}\,\mathrm{du} = \int a\,\mathrm{db} = ab - \int b\,\mathrm{da} = \left.\frac{e^u}{u+1}\right|_0^1 + \int_0^1 \frac{e^u}{(u+1)^2}$$

Simplifying, we have the following.

$$\left.\frac{e^u}{u+1}\right|_0^1 + \int_0^1 \frac{e^u}{(u+1)^2} = \left(\frac{e^1}{1+1}\right) - \left(\frac{e^0}{0+1}\right) + \int_0^1 \frac{e^u}{(u+1)^2}\,\mathrm{du} = \frac{e}{2} - 1 + \int_0^1 \frac{e^u}{(u+1)^2}\,\mathrm{du}$$

Substituting this equivalent expression, we note that the two integrals cancel, and we are left with the final result.

$$I = \int_0^1 \frac{u}{(u+1)^2}\cdot(e^u\,\mathrm{du}) = \left(\frac{e}{2} - 1 + \int_0^1 \frac{e^u}{(u+1)^2}\,\mathrm{du}\right) - \int_0^1 \frac{e^u}{(u+1)^2}\,\mathrm{du} = \boxed{\frac{e}{2}-1}$$

Problem 14. $\int_0^1 \left\lfloor \sin\left(\frac{\pi}{x}\right)\right\rfloor\,\mathrm{dx}$

Proposed by Jeck Lim

Solution: $\boxed{\ln(1/2)}$ **or** $\boxed{-\ln(2)}$

If we pick any $x \in [1/2, 1]$, we note the following occurs at the bounds.

$$\sin\left(\frac{\pi}{\frac{1}{2}}\right) = \sin(2\pi) = 0 \qquad \sin\left(\frac{\pi}{1}\right) = \sin(\pi) = 0$$

We now need to note if the signed area will be positive or negative. Between the angles of π and 2π, $\sin(x)$ will be negative, namely less than 0. Due to the floor function, this rounds down and we have that it evaluates to -1. Thus, on this interval of $[0.5, 1]$, the function in question will have a rectangle with width 0.5, as well as a height of -1 (this implicates that the rectangle is subtended **underneath** the x-axis).

Repeating this process on the interval of $x \in [1/3, 1/2]$, we note that the following occurs at the bounds.

$$\sin\left(\frac{\pi}{\frac{1}{3}}\right) = \sin(3\pi) = 0 \qquad \sin\left(\frac{\pi}{2}\right) = \sin(2\pi) = 0$$

We now need to note if the signed area will be positive or negative. Between the angles of 2π and 3π, $\sin(x)$ will be positive, namely greater than 0. Due to the floor function, this rounds down and we have that it evaluates to 0. Thus, on this interval of $[0.5, 1]$, the function in question will have a rectangle with width 0.5 and a height of 0.

We can now see the general pattern. For any interval $[1/k, 1/(k+1)]$, we have that if k is odd the function evaluates to 0. If we instead have that k is even, we have that the function evaluates to -1. The function only gives the height of the rectangle, which means the width of the rectangle can be calculated from the length of the interval, which is $1/k - 1/(k+1)$. Thus, the area summation is as follows.

$$\int_0^1 \left\lfloor \sin\left(\frac{\pi}{x}\right) \right\rfloor \,\mathrm{dx} = A = \sum_{k=1}^{\infty} (-1) \cdot \left(\frac{1}{k} - \frac{1}{k+1}\right) = -\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right)$$

We can recognize this as the alternating harmonic series, which converges as follows.

$$A = -\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right) = -(\ln(2)) = \boxed{\ln(1/2)} = \ln\left(2^{-1}\right) = \boxed{-\ln(2)}$$