Chapter 5

Complex Numbers

5.1 The Set of Complex Numbers

In a previous chapter, we noted that, for any real number $x \in \mathbb{R}$, it is always true that $x^2 \geq 0$. Thus, solutions to equations like $x^2 = -1$ are never possible when considering only real numbers. This may seem as if it's not a big deal, until we realize that this prohibits us from finding roots to a polynomial as simple $x^2 + 1$. Since roots of polynomials are crucial in various fields of Mathematics, mathematicians wished to expand their notion of a number to include solutions to $x^2 + 1 = 0$. In doing so, the complex numbers were created and Mathematics as we know it has never been the same. This chapter will provide an introduction to the basic algebraic structure of the complex numbers, along with some interesting applications.

5.1.1 Constructing Complex Numbers.

Since no real solution to the equation $x^2 + 1 = 0$ exists, mathematicians simply created an object that had this property and called it the imaginary number i. Thus, i is a non-real number such that $i^2 = -1$. To make i relevant to the existing set of real numbers $\mathbb{R}$, mathematicians needed to place this i in a set that seemed to be larger than $\mathbb{R}$ itself but still followed many of the same algebraic rules as $\mathbb{R}$.

Thus, we defined the set of complex numbers, denoted by $\mathbb{C}$, as

$$\mathbb{C} = \{a + bi \,|\, a, b \in \mathbb{R}\}.$$

An individual complex number is simply a real number a added to another real number b multiplied by this new imaginary number i. From its definition, it is clear that $\mathbb{C}$ seems to contain two copies of the real numbers $\mathbb{R}$; one of which stands by itself and the other of which is attached to i. If we write $z = a + bi$, then we have the following definitions.

- The **real part** of z is the portion of the complex number not attached to the i. Thus, it is given by

  $$\text{Re}(z) = \text{Re}(a + bi) = a.$$

- The **imaginary part** of z is the portion of the complex number that is attached to the i. Thus, it is given by

  $$\text{Im}(z) = \text{Im}(a + bi) = b.$$

  In particular, the imaginary part *does not* include the imaginary i term.

It is important to note that if z is a complex number, then its real and imaginary parts are both real numbers.

5.1.2 The Reals as a Subset of the Complex Numbers

Since the complex numbers were seen as an extension of the set of real numbers, it is natural to believe that $\mathbb{R}$ is a subset of $\mathbb{C}$. Of course, to prove this subset inclusion, we must show that if $x \in \mathbb{R}$, then $x \in \mathbb{C}$. Indeed, if $x \in \mathbb{R}$, then $x = x + 0i \in \mathbb{C}$. Thus, $\mathbb{R} \subset \mathbb{C}$. In fact, we have the following series of subset inclusions:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

5.2 Complex Arithmetic

Above, we showed that $\mathbb{C}$ is an extension of $\mathbb{R}$ in the sense that $\mathbb{R}$ is a subset of $\mathbb{C}$. We must, however, also endow $\mathbb{C}$ with an algebraic structure that is compatible with the algebraic structure on $\mathbb{R}$. Indeed, complex arithmetic is defined exactly as one would expect, with the added condition that $i^2 = -1$. If $z = a + bi$ and $w = c + di$, we can define *complex arithmetic* in the following way.

- *Complex Addition.* To add $z = a + bi$ and $w = c + di$, we simply combine their real and imaginary parts to obtain

$$z + w = (a + bi) + (c + di) = (a + c) + (b + d)i.$$

- *Complex Multiplication.* To multiply $z = a + bi$ and $w = c + di$, we simply FOIL, use the fact that $i^2 = -1$, and combine real and imaginary parts to obtain

$$z \cdot w = (a + bi) \cdot (c + di) =$$
$$ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

- *Complex Inversion.* If $z = a + bi$ and $z \neq 0 + 0i$, then we can find $\frac{1}{z}$ by multiplying by a special form of 1 that is given by (what will soon be called) the *complex conjugate*:

$$\frac{1}{z} = \frac{1}{a + bi} = \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} =$$
$$\frac{a - bi}{a^2 - b^2 i^2} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2} i.$$

  Notice that, since $z \neq 0 + 0i$, then $a \neq 0$ or $b \neq 0$ so the denominators $a^2 + b^2 \neq 0$.

- *Complex Division.* Since we know how to multiply and how to invert, we can define complex division $z \div w$ for $w \neq 0$ as

$$z \div w = z \cdot \frac{1}{w}.$$

The above arithmetic operations on $\mathbb{C}$ give it an algebraic structure that is very similar to $\mathbb{R}$ but also utilizes the fact that $i^2 = -1$. Notice that the above definitions also show that $\mathbb{C}$ enjoys nice algebraic properties like being closed under addition, multiplication, additive inverses, and multiplicative inverses. In fact, $\mathbb{C}$, just like $\mathbb{Q}$ and $\mathbb{R}$, is a *field*.

5.2.1 Complex Conjugation

Above, we saw that we were able to write the inverse $\frac{1}{a+bi}$ in standard complex number form by multiplying by a special fraction equal to 1, where the numerator and denominator were both equal to $a - bi$. The relationship between $a + bi$ and $a - bi$ is called *complex conjugation*. More precisely, if $z = a + bi$, we define its **complex conjugate** $\overline{z}$ to be

$$\overline{z} = \overline{a + bi} = a - bi.$$

Thus, conjugation leaves the real part of a complex number alone and negates its imaginary part.

Complex conjugation is a very important operation on the set of complex numbers. Below are some of its more helpful features.

- Performing complex conjugation twice returns the original input. In other words, if $z = a + bi$, then

$$\overline{\overline{z}} = \overline{a - bi} = a + bi = z.$$

- We can use the complex conjugate of a number to define the real part of a complex number. Namely,

$$\text{Re}(z) = \frac{z + \overline{z}}{2}.$$

  To verify this, we see that

$$\frac{z + \overline{z}}{2} = \frac{a + bi + \overline{a + bi}}{2} = \frac{a + bi + a - bi}{2} = \frac{2a}{2} = a = \text{Re}(z).$$

- Similarly, we can use the complex conjugate of a number to define the imaginary part of a complex number. Namely,

$$\text{Im}(z) = \frac{z - \overline{z}}{2i}.$$

  To verify this, we see that

$$\frac{z - \overline{z}}{2i} = \frac{a + bi - \overline{(a + bi)}}{2i} = \frac{a + bi - (a - bi)}{2i} = \frac{2bi}{2i} = b = \text{Im}(z)$$

- In numerous instances, the number $z \cdot \overline{z}$ is frequently utilized. Notice that

$$z \cdot \overline{z} = (a + bi)(a - bi) = a^2 + b^2.$$

  Thus, $z \cdot \overline{z}$ is a real, non-negative number. In fact, $z \cdot \overline{z}$ is 0 if and only if $z = 0 + 0i$.

In fact, one of the most helpful aspects of the complex conjugate is to test if a complex number $z = a + bi$ is real. A complex number is real if and only if $z = a + 0i$; in other words, a complex number is real if it has an imaginary part of 0.

Proposition. Let $z \in \mathbb{C}$. z is real if and only if $z = \overline{z}$.

Discussion. Our statement is the biconditional $p \Leftrightarrow q$ where p is given by "z is real" and q is given by "$z = \overline{z}$." Thus, we need to prove the following two conditional statements:

- $p \Rightarrow q$: "If z is real, then $z = \overline{z}$," which will be a direct calculation.
- $q \Rightarrow p$: "If $z = \overline{z}$, then z is real," which we can show by showing that $z = a + 0i$.

Proof. Let $z = a+bi$, where $a, b \in \mathbb{R}$. We will prove our biconditional statement by showing two conditional statements: "If z is real, then $z = \overline{z}$" and "If $z = \overline{z}$, then z is real."

For the first conditional statement, assume that z is real. Then, $z = a + 0i$. Thus,

$$\overline{z} = \overline{a + 0i} = a - 0i = a = z,$$

as desired.

For the second conditional statement, assume that $z = \overline{z}$. Then,

$$a + bi = \overline{a + bi} = a - bi,$$

and thus $a + bi = a - bi$. Subtracting a from both sides, we get that $bi = -bi$. Dividing by i, we get $b = -b$ and thus $2b = 0$. Thus, $b = 0$ and so $z = a + bi = a + 0i$, a real number.

Having proven both conditional statements, we have shown that "z is real if and only if $z = \overline{z}$."

□

5.3 The Geometry of the Complex Plane

5.3.1 The Complex Plane

From its definition, the set of complex numbers $\mathbb{C}$ can be seen as something resembling the product of two copies of $\mathbb{R}$, one for its real part and the other for the imaginary part. This implies that, if we are to assign some kind of geometry to the complex numbers, that we should look at the real plane $\mathbb{R}^2$ for inspiration.

The real plane $\mathbb{R}^2$ is given by

$$\mathbb{R}^2 = \{(x, y) \,|\, x, y \in \mathbb{R}\}.$$

Geometrically, we interpret this as a two-dimensional plane, where individual elements of the plane can be given in terms of their x and y coordinates. Since the complex numbers are defined as

$$\mathbb{C} = \{a + bi \,|\, a, b \in \mathbb{R}\},$$

we can think of the real part a of the complex number as the x-coordinate and the imaginary part b as the y-coordinate. Thus, the **complex plane** is the two-dimensional plane with two axes, a horizontal *real axis* and a vertical *imaginary axis*. We position the complex number $a + bi$ to have coordinate a on the real axis and b on the imaginary axis. Thus, if we were to think of the complex plane as $\mathbb{R}^2$, then the complex number $a + bi$ should be thought of as (a, b).

We can further interpret some of the above algebra of the complex numbers in this new geometric lens. We give a couple of examples below.

- If $z, w \in \mathbb{C}$, then the sum $z + w$ is the point on the plane that, along with z, w, and 0, are the vertices of a parallelogram.
- If $z \in \mathbb{C}$, then its complex conjugate $\overline{z}$ is simply the reflection of the point z about the real axis. Notice that if z is real, then it lies on the real axis and thus reflecting about that axis will not change the point; this is the geometric manifestation of the fact that if z is real, then $\overline{z} = z$.

5.3.2 The Modulus of a Complex Number

In the regular geometry on $\mathbb{R}^2$, we can compute the distance between $(x, y) \in \mathbb{R}^2$ and the origin $(0, 0)$ by

$$\sqrt{x^2 + y^2}.$$

If we translate this in terms of the complex plane, then the distance between a complex number $z = a + bi$ and the complex number $0 + 0i$ is

$$\sqrt{a^2 + b^2}.$$

This inspires the following definition: if $z = a + bi \in \mathbb{C}$, then we can define its **modulus** by

$$|z| = \sqrt{a^2 + b^2}.$$

Notice that, as we expect with a "distance", even though z is complex, $|z|$ is always real and non-negative. In fact, it enjoys many other properties, given below.

- z is equal to 0 if and only if $|z| = 0$. This is true because $\sqrt{a^2 + b^2} = 0$ only if both a and b are zero.

- We can write the modulus in terms of complex conjugation by

$$|z| = \sqrt{z \cdot \overline{z}}.$$

- The notation $|z|$ is, of course, identical to that of the *absolute value* of a real number. Indeed, if $z = a + 0i$ is real, then

$$|z| = |a + 0i| = \sqrt{a^2 + 0^2} = |a|,$$

  where $|z|$ means the modulus of the complex number z and $|a|$ means the absolute value of the real number a. Thus, the complex modulus is a generalization of the absolute value of a real number.

- $|z|$ is seen as the distance in the complex plane between z and $0+0i$ (which is at the intersection of the real and imaginary axes). We can, however, compute the distance between any two complex number $z, w \in \mathbb{C}$ by

$$|z - w|.$$

- We can use the modulus to define interesting curves in the complex plane. For example, if we wish to sketch the curve associated to $|z| = r$ for some real $r \geq 0$, then we are asking for all complex numbers z with the property that their modulus (distance to the origin) is equal to r. Writing this out, if $z = a + bi$, then $|z| = r$ is equivalent to

$$\sqrt{a^2 + b^2} = r.$$

  Squaring both sides, we get the equation $a^2 + b^2 = r^2$. Since a and b give the real and imaginary coordinates, this simply corresponds to a circle of radius r about the origin.

- Generalizing the equation above, we can obtain the circle of radius $r \geq 0$ centered at the complex number $w \in \mathbb{C}$ by considering all solutions z to the equation $|z - w| = r$. Since $|z - w|$ is the distance from z to w, then this equation asks for all points z a distance of r from w, which is precisely a circle of radius r about w.

5.4 Euler's Equation and the Polar Representation of the Complex Plane

5.4.1 From Taylor Series to the Euler Equation

In Calculus, we were exposed to the Taylor Series for several important real functions like e^x, $\cos x$, and $\sin x$. They are given below

$$\begin{aligned} e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\ \cos x &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \\ \sin x &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}. \end{aligned}$$

Many students comment on the fact that these series have similar characteristics. For example, all contain factorial terms in the denominator and, more precisely, the term being factorialized is equal to the exponent of x in the numerator. One big difference, though, is that e^x contains all natural numbers as exponents while $\cos x$ and $\sin x$ have only even and odd exponents, respectively.

This last observation leads to the following question: Is there a way to write the Taylor expansion for e^x in terms of the Taylor expansions for $\cos x$ and $\sin x$? Since $\cos x$ and $\sin x$ have the odd and even exponents, perhaps they can be added up in a way that would produce all of the Taylor expansion for e^x.

Given our newfound knowledge of imaginary numbers, we can ask what happens when we plug in $x = i\theta$ into the Taylor expansion for e^x. To do so, we will first note that raising i to whole number exponents will produce the following cyclic behavior:

$$i^1 = i, \quad i^2 = -1, \quad i^3 = -i, \quad \text{and} \quad i^4 = 1,$$

which will repeat over and over. Thus, we can expand out $e^{i\theta}$ as

$$e^{i\theta} = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} =$$

$$1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \frac{(i\theta)^8}{8!} + \cdots =$$

$$1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \frac{\theta^8}{8!} + \cdots =$$

$$\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) =$$

$$\cos\theta + i\sin\theta.$$

Thus, we have arrived at one of the most important mathematical equations ever produced. It is called the **Euler equation** and it says that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Before we begin discussing the deep importance of this equation, we must note that in the above computation, we moved around the order of infinitely many terms in the Taylor series. This cannot be done in general but is allowed when our series are *absolutely convergent*, which all of ours are.

5.4.2 The Euler Equation and Complex Polar Form

The utility of the Euler equation comes from the fact that its right-hand side $\cos\theta + i\sin\theta$ is a complex number with real part $\cos\theta$ and imaginary part $\sin\theta$. Since this complex number is equal to $e^{i\theta}$, then any complex number of the form $\cos\theta + i\sin\theta$ can be replaced by $e^{i\theta}$.

If we think about which complex numbers have their real part being $\cos\theta$ and their imaginary parts being $\sin\theta$, we can find motivation in asking which points in $\mathbb{R}^2$ have their x-values being $\cos\theta$ and their y-values being $\sin\theta$. Using basic trigonometry, we can see that a point on the circle of radius 1 centered at the origin making a counterclockwise angle of θ with the positive x-axis has exactly these coordinates.

If we have a point that is not on the circle of radius 1 centered at the origin, then it is on the circle of radius r centered at the origin for some $r \geq 0$. More basic trigonometry tells us that a point on a circle of radius r about the origin making an angle of θ with the positive x-axis has x-coordinate $r\cos\theta$ and y-coordinate $r\sin\theta$. In complex notation, this is just $r\cos\theta + ir\sin\theta$. By the Euler equation, though, we have that

$$re^{i\theta} = r\cos\theta + ir\sin\theta.$$

Thus, we have arrived at a new representation for complex numbers that is almost identical to that of polar coordinates. This new polar form is given as follows: the complex number $re^{i\theta}$ is the point on the complex plane that is distance r from the origin and makes a counterclockwise angle of θ with the positive real axis.

Below are some examples and properties of this new representation for complex numbers.

- The complex number $3e^{i\frac{\pi}{3}}$ can be computed using Euler's equation by

$$3e^{i\frac{\pi}{3}} = 3\cos\left(\frac{\pi}{3}\right) + i3\sin\left(\frac{\pi}{3}\right) =$$

$$3\frac{1}{2} + i3\frac{\sqrt{3}}{2} = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

- Notice that the complex number i makes an angle of $\pi/2$ with the positive real axis and is distance 1 form the origin. Thus,

$$i = e^{i\frac{\pi}{2}}.$$

  We can confirm this using Euler's equation:

$$e^{i\frac{\pi}{2}} = \cos(\pi/2) + i\sin(\pi/2) = 0 + 1i = i.$$

- Notice that any positive real number $r \geq 0$ makes an angle of 0 with the positive real axis and is distance r from the origin. Thus,

$$r = re^{i0},$$

  which is certainly true since $e^0 = 1$.

- The real number -1 makes an angle of π with the positive real axis and is distance 1 from the origin. Thus,

$$-1 = e^{i\pi}.$$

  This is remarkable because, if we slightly rearrange this expression, we arrive at an equation that includes the five most important mathematical constants $(0, 1, e, i, \pi)$:

$$e^{i\pi} + 1 = 0.$$

- Unlike with the standard (Cartesian) form for a complex number $a + bi$, the polar form $re^{i\theta}$ is non-unique. For example, if we replace θ with $\theta + 2\pi$ (or add any other integer multiple of 2π), the complex number does not change. Thus,
$$re^{i\theta} = re^{i(\theta+2\pi k)}$$
for any $k \in \mathbb{Z}$. Algebraically, this comes from the fact that both $\sin x$ and $\cos x$ are periodic of period 2π. Geometrically, it comes form the fact that making an angle of θ is the same as making an angle of $\theta + 2\pi k$.

5.4.3 The Algebra & Geometry Associated to $re^{i\theta}$

The polar form of a complex number will make many computations significantly easier. Unlike the cartesian $a + bi$ form, the polar form seems to be better suited for dealing with complex multiplication. Before we begin, we note that, indeed, the algebra associated to having a complex exponent is identical to the algebra associated to having a real exponent. In particular, all of the usual laws of exponents hold. Using this, we have the following algebraic and geometric properties of $re^{i\theta}$.

- Complex multiplication can be more easily seen using the polar form. In particular, if we have two complex numbers z_1 and z_2 written in polar form, then $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$. Multiplying, we have
$$z_1 \cdot z_2 = r_1e^{i\theta_1} \cdot r_2e^{i\theta_2} = (r_1r_2)e^{i(\theta_1+\theta_2)}.$$
Thus multiplying two complex numbers $r_1e^{i\theta_1}$ and $r_2e^{i\theta_2}$ will produce another complex number that is distance r_1r_2 away from the origin and makes an angle of $\theta_1 + \theta_2$ with the positive real axis.

- We expect that the modulus of a complex number $z = re^{i\theta}$ will be equal to r, since this is its distance to the origin. Using Euler's equation, we see that this is indeed true:
$$|z| = \left|re^{i\theta}\right| = |r\cos\theta + ir\sin\theta| =$$
$$\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = r\sqrt{\cos^2\theta + \sin^2\theta} = r.$$
Here, we used the well-known Pythagorean identity that $\sin^2\theta + \cos^2\theta = 1$ for all values of θ.

- We can also use the Euler equation to compute the conjugate of a complex number $z = re^{i\theta}$ in polar form. Recalling that $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$, we have that
$$\overline{z} = \overline{re^{i\theta}} = \overline{r\cos\theta + ir\sin\theta} =$$
$$r\cos\theta - ir\sin\theta = r\cos(-\theta) + ir\sin(-\theta) = re^{i(-\theta)}.$$
Thus, the conjugate of $re^{i\theta}$ is just $re^{i(-\theta)}$; so, taking a conjugate will keep the distance to the origin the same, but will making an angle of $-\theta$ instead of θ. This geometrically corresponds to reflecting the point about the real axis, as is true with the usual Cartesian form of complex numbers.

- We can raise e to any complex number. For example, if we have $a + bi$, then we place this in the exponent and simplify using basic exponent rules:
$$e^{a+bi} = e^a \cdot e^{ib}.$$
Since $a, b \in \mathbb{R}$, then e^{a+bi} will be a distance e^a from the origin and will make an angle of b with the positive real axis.

Euler's equation also provides us with an excellent way to remember the various trigonometric identities that we use frequently in Mathematics. For example, consider the following.

- Consider $e^{i(2\theta)}$. Since complex exponents follow the same rules as real exponents, then we can re-write $e^{i(2\theta)}$ as $\left(e^{i\theta}\right)^2$. Expanding the $e^{i(2\theta)}$ gives us

  $$e^{i(2\theta)} = \cos(2\theta) + i\sin(2\theta).$$

  Expanding $\left(e^{i\theta}\right)^2$ gives us

  $$\left(e^{i\theta}\right)^2 = (\cos\theta + i\sin\theta)^2 =$$

  $$\cos^2\theta + 2i\sin\theta\cos\theta + i^2\sin^2\theta = (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta).$$

  Since the above expressions are all equal, then

  $$\cos(2\theta) + i\sin(2\theta) = (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta).$$

  For two complex numbers to be equal, their real parts and their imaginary parts must be equal. Equating these, we obtain two well-known double-angle formulae from trigonometry:

  $$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

  $$\sin(2\theta) = 2\sin\theta\cos\theta.$$

- We may generalize the above observations by using the more general fact that $e^{i(n\theta)} = (e^{i\theta})^n$. Re-writing this using the Euler equation, we get **DeMoirve's Formula**, which states that

  $$\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n.$$