CHMMC 2020-2021

Proof Round Solutions

1. *(5 pts)* Let n be a positive integer, $K = \{1, 2, \ldots, n\}$, and $\sigma : K \to K$ be a function with the property that $\sigma(i) = \sigma(j)$ if and only if $i = j$ (in other words, σ is a *bijection*). Show that there is a positive integer m such that

$$\underbrace{\sigma(\sigma(\ldots\sigma(i)\ldots))}_{m \text{ times}} = i$$

for all $i \in K$.

Solution:

Denote σ^k as the composition of σ by k times. Consider any $i \in K$. Assume on the contrary that the following sequence

$$\sigma(i), \sigma^2(i), \ldots \sigma^n(i)$$

does not contain i. Then, by the Pigeonhole Principle, there exists some $1 \le a < b \le n$ so that $\sigma^a(i) = \sigma^b(i)$. Since σ is a bijection, the pre-images of $\sigma^a(i)$ and $\sigma^b(i)$ are unique and given by $\sigma^{a-1}(i)$ and $\sigma^{b-1}(i)$, respectively. Furthermore, since $\sigma^a(i) = \sigma^b(i)$, we have that $\sigma^{a-1}(i) = \sigma^{b-1}(i)$. Repeating this analysis, we have that $i = \sigma^{a-a}(i) = \sigma^{b-a}(i)$, a contradiction. Thus, our assumption is false, so there must exist a positive integer $m_i \le n$ such that $\sigma^{m_i}(i) = i$.

Hence, the value $m = \operatorname{lcm}(m_1, \ldots, m_n)$ satisfies the desired property. $\square$

2. *(5 pts)* For some positive integer n, let $P(x)$ be an nth degree polynomial with real coefficients.

Note: you may cite, without proof, the Fundamental Theorem of Algebra, which states that every non-constant polynomial with complex coefficients has a complex root.

(a) *(2 pts)* Show that there is an integer $k \ge \frac{n}{2}$ and a sequence of non-constant polynomials with real coefficients $Q_1(x), Q_2(x), \ldots, Q_k(x)$ such that

$$P(x) = \prod_{i=1}^{k} Q_i(x).$$

(b) *(1 pt)* If n is odd, then show that $P(x)$ has a real root.

(c) *(2 pts)* Let a and b be real numbers, and let m be a positive integer. If $\zeta = a + bi$ is a nonreal root of $P(x)$ of multiplicity m, then show that $\overline{\zeta} = a - bi$ is a nonreal root of $P(x)$ of multiplicity m.

Solution (a):

We proceed with strong induction on n and show that we can take $k \ge \lceil \frac{n}{2} \rceil$.

Base case: $n = 1, 2$. By assumption, a linear or quadratic polynomial with real coefficients can be written as the product of $\lceil \frac{n}{2} \rceil = 1$ polynomial with real coefficients. This verifies the base case.

Induction step: $n \ge 3$. By the Fundamental Theorem of Algebra, $P(x)$ has a complex root ζ.

If ζ is real, then we see that $P(x) = (x - \zeta)R_1(x)$ where $R_1(x)$ is degree $n-1$ with real coefficients. By the inductive assumption, $R_1(x)$ can be written as a product of $\lceil \frac{n-1}{2} \rceil$ polynomials with real coefficients. Since $x - \zeta$ has real coefficients, we conclude that $P(x)$ can be written as a product of $\lceil \frac{n-1}{2} \rceil + 1 \ge \lceil \frac{n}{2} \rceil$ polynomials with real coefficients.

On the other hand, if ζ is nonreal, then by the Conjugate Root Theorem, $\overline{\zeta}$ is a nonreal root of $P(x)$ as well. Then, $(x-\zeta)(x-\overline{\zeta}) = x^2 - (\zeta+\overline{\zeta})x + |\zeta|^2$ has real coefficients and in fact $P(x) = (x-\zeta)(x-\overline{\zeta})R_2(x)$ where $R_2(x)$ is degree $n-2$ with real coefficients. By the inductive assumption, $R_2(x)$ can be written as a product of $\lceil \frac{n-2}{2} \rceil$ polynomials with real coefficients. We conclude that $P(x)$ can be written as a product of $\lceil \frac{n-2}{2} \rceil + 2 = \lceil \frac{n}{2} \rceil$ polynomials with real coefficients. This completes the induction and thus the proof. □

Solution (b):

By (a), we have that

$$P(x) = \prod_{i=1}^{k} Q_i(x).$$

where $k \geq \frac{n}{2}$. Since n is odd, we see in fact that $k > \frac{n}{2}$. By the Pigeonhole Principle, there must be some $Q_j(x) = ax - b$, a degree 1 polynomial with real coefficients. $Q_j(x)$ has a real root $\frac{b}{a}$, so $P(x)$ does as well. □

Solution (c):

Suppose that ζ has multiplicity m and $\overline{\zeta}$ has multiplicity s, where s is a positive integer. Then we see that $P(x) = (x-\zeta)^m(x-\overline{\zeta})^s R(x)$, where $\zeta, \overline{\zeta}$ are not roots of polynomial $R(x)$.

Assume on the contrary that $s < m$. Then $P(x) = (x-\zeta)^s(x-\overline{\zeta})^s(x-\zeta)^{m-s}R(x)$. Since $(x-\zeta)^s(x-\overline{\zeta})^s = (x^2 - (\zeta+\overline{\zeta})x + |\zeta|^2)^s$ has real coefficients, we see that $(x-\zeta)^{m-s}R(x)$ has real coefficients as well. Since ζ is a root of $(x-\zeta)^{m-s}R(x)$, $\overline{\zeta}$ is a root of $(x-\zeta)^{m-s}R(x)$ as well. This is a contradiction by the earlier assumption that $\zeta, \overline{\zeta}$ are not roots of polynomial $R(x)$.

We can disprove the case $s > m$ by a similar argument. Hence, we conclude that $s = m$. □

3. *(6 pts)* Find all positive integers $n \geq 3$ such that there exists a permutation $a_1, a_2, \ldots, a_n$ of $1, 2, \ldots, n$ such that $a_1, 2a_2, \ldots, na_n$ can be rearranged into an arithmetic progression.

Solution:

We claim that $a_1, 2a_2, \ldots, na_n$ cannot be arranged as an arithmetic progression for all integers $n \geq 3$.

Lemma: For a prime p and an arithmetic progression of integers $A = x_1, x_2, \ldots, x_k$ with common difference d such that $p \nmid d$, there will be $k_1 \in \{\lfloor \frac{k}{p} \rfloor, \lceil \frac{k}{p} \rceil\}$ terms of the progression that are multiples of p. Furthermore, at most $\lceil \frac{k_1}{p} \rceil$ of those multiples of p are also multiples of p^2. *Note: if $p \mid d$, then all elements a_i are mutually congruent $\pmod{p}$, so there must be either 0 or k multiples of p in the arithmetic progression. Also, note this arithmetic progression is an arbitrary one, not the one originally referred to in the problem.*

Proof of lemma: If $p \nmid d$, then by Bezout's Lemma there must be some integer $1 \leq j \leq p$ such that $p \mid x_1 + d(j-1) = x_j$. Therefore, the multiples of p in A are precisely indexed by the set of integers $\{j + p\mathbb{Z}\} \cap \{1, 2, \ldots, k\}$. The multiples of p in A occur in intervals of p terms, so A contains either $\lfloor \frac{k}{p} \rfloor$ or $\lceil \frac{k}{p} \rceil$ multiples of p.

Again using the notation from the **Lemma**, let B be the arithmetic subsequence of A containing only the $k_1 \in \{\lfloor \frac{k}{p} \rfloor, \lceil \frac{k}{p} \rceil\}$ multiples of p. B has common difference pd, so we may divide all terms in B by p to obtain another arithmetic sequence $\frac{B}{p}$ of integers with common difference d. However, $p \nmid d$, so by the argument from the above paragraph, $\frac{B}{p}$ contains at most $\lceil \frac{k_1}{p} \rceil$ multiples of p. It follows that B contains at most $\lceil \frac{k_1}{p} \rceil$ multiples of p^2.

Now return to the problem. We use the above observations repeatedly to address the following cases of n.

n is prime: If $a_1, 2a_2, \cdots, na_n$ could be arranged as an arithmetic progression, then by the Lemma, we would either have exactly one term ja_j to be divisible by n or all terms ja_j to be divisible by n (there is clearly at least one multiple of n). Since n is prime, at most two terms of $a_1, 2a_2, \cdots, na_n$ are multiples of n. Thus, it is impossible for all terms ja_j to be divisible by n. Therefore the former case must hold; since n is prime, it follows that $a_n = n \implies na_n = n^2$. The arithmetic progression has common difference $\geq 2n+1$, as the second largest possible term is at most $(n-1)^2$. Then the smallest term of the arithmetic progression is at most $n^2 - (n-1)(2n+1) \leq 0$, a contradiction.

$n = 4, 6, 8, 10$: If $a_1, 2a_2, \cdots, na_n$ could be arranged as an arithmetic progression, then by the Lemma, we would either have exactly $\frac{n}{2}$ terms ja_j to be even or all terms ja_j to be even. If exactly half of the terms ja_j is even, then they are also multiples of 4, since n is even, which is impossible by the Lemma. If all terms ja_j are even, then we are "pairing" each even number in $1, 2, \ldots, n$ to an odd number, and each odd number in $1, 2, \ldots, n$ to an even number. Now consider the following two subcases:

- $n = 4, 8$: The sequence $a_1, 2a_2, \cdots, na_n$ must have multiples of 4 and non-multiples of 4; if all the terms were multiples of 4 then every odd $j \in \{1, \ldots, n\}$ would be paired with a $0 \pmod 4$ element of $\{1, \ldots, n\}$, which is not possible. Thus, the common difference d is even but not a multiple of 4. Suppose i is such that $a_i = n$; we pair evens and odds, so $i \neq n$. Then, we see that na_n and ia_i differ by a multiple of $2n = 8, 16$; both are divisible by n and n is even and we pair evens to odds so a_n and i are both odd. This is not possible: since $4 \nmid d$, the common difference between any two terms of $a_1, 2a_2, \cdots, na_n$ cannot be a multiple of 8 (for the case $n = 4$) or 16 (for the case $n = 8$). In particular, if two terms na_n, ia_i are $n > k > 0$ terms apart in the arithmetic progression, then $dk = 2n$, so k, n have the same number of factors of 2. But n is a power of 2 in these cases, so $k \geq n$, a contradiction.
- $n = 6, 10$: as we pair odd and even numbers in $a_1, 2a_2, \cdots, na_n$, we see that there are exactly $2, 4$ values ja_j that are multiples of 4, respectively. By the Lemma, both of these cases are impossible.

$n = 9, n \geq 12$: We may write $n = 3k + m$, where $k \geq 3, m = 0, 1, 2$. If $a_1, 2a_2, \cdots, na_n$ could be arranged as an arithmetic progression, then by the Lemma, we would either have k terms ja_j to be multiples of 3, $k+1$ terms ja_j to be multiples of 3 (only if $m \neq 0$), or all terms ja_j to be multiples of 3. The last case is clearly impossible since we can "distribute" multiples of 3 to at most $2k$ terms ja_j in the sequence; exactly k of $\{1, \ldots, n\} = \{a_1, \ldots, a_n\}$ are divisible by 3. If there are only k values ja_j that are multiples of 3, then they are also multiples of 9 (since we multiply $3, 6, \ldots, 3k$ by multiples of 3: $a_3, a_6, \ldots, a_{3k}$, there being exactly k of $\{1, \ldots, n\} = \{a_1, \ldots, a_n\}$ are divisible by 3). By the Lemma, this is impossible, as $\lceil \frac{k}{3} \rceil < k$.

Finally, if there are $k+1$ terms ja_j that are multiples of 3, then we know that there are still at least $k-1$ of those terms that are also multiples of 9, there being exactly k of $\{1, \ldots, n\} = \{a_1, \ldots, a_n\}$ are divisible by 3. However, by the Lemma there can be at most $\lceil \frac{k+1}{3} \rceil$ values ja_j that are multiples of 9, implying that $\lceil \frac{k+1}{3} \rceil < k - 1$ for all $k \geq 4$, which is a contradiction.

We have covered all possible values of $n \geq 3$, thus completing the proof. $\square$

4. *(7 pts)* Fix a positive integer n. Pick $4n$ equally spaced points on a circle and color them alternately blue and red. You use n blue chords to pair the $2n$ blue points, and you use n red chords to pair the $2n$ red points. If some blue chord intersects some other red chord, then such a pair of chords is called a "good pair."

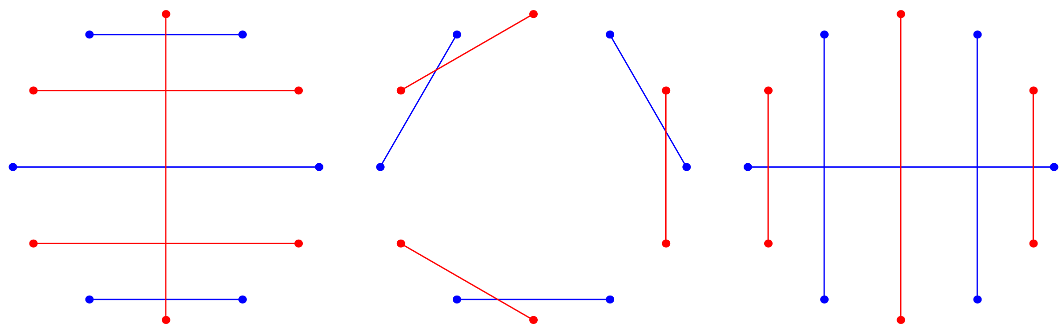
(a) *(1 pts)* For the case $n = 3$, explicitly show that there are at least 3 distinct ways to pair the $2n$

blue points and the $2n$ red points such that there are a total of 3 good pairs (2 configurations of chord pairings are *not* considered distinct if one of them can be "rotated" to the other).

(b) *(6 pts)* Now suppose that n is arbitrary. Find, with proof, the minimum number of good pairs under all possible configurations of chord pairings.

Solution (a):

Here are 3 possible ways to obtain 3 good pairs in the case $n = 3$:



Solution (b):

We prove a Lemma beforehand:

Lemma: Given $2n$ fixed points on a circle. We connect these $2n$ points with n chords, such that each point is connected to exactly one chord. Consider doing the following operations: For any two chords AC and BD that intersect with each other inside the circle (assume that the four points A, B, C, D are counter-clockwisely oriented on the circle), we replace the two chords with AB and CD. Such operation must terminate after finitely many steps.

Proof of lemma: Let k denote the number of ways to select n chords connecting $2n$ fixed points on a circle, such that each point is connected to exactly one chord. Given that $n < \infty$, we have that k must also be finite. For any $1 \leq i \leq k$, consider the i-th way of selecting such n chords. Correspondingly, let L_i denote the sum of the lengths of the n chords selected in the i-th way. Let $S = \{L_1, L_2, \cdots, L_k\}$ denote the set of all sums of the n chords' lengths. We have that S is also a finite set. Now we consider the operation described in the Lemma.

Assume that the intersection of the two chords AC and BD is O, which is in the interior of the circle. Then we have that $AO + BO > AB$ and $CO + DO > CD$. Taking the sum of the two inequalities yields

$$AC + BD = (AO + CO) + (BO + DO) = (AO + BO) + (CO + DO) > AB + CD.$$

Hence, we have that after such operation, the variable L, which denotes the sum of the n selected chords' lengths, must strictly decrease. Since we always have $L \in S$ and S is a finite set, we can conclude that such operation must terminate after finitely many steps (when L attains the minimum element of S).

Now return to the original problem.

Color the intersection of a red chord and a blue chord green. Then we have that the number of green points is equal to the number of "good pairs". Now consider applying the operation described in the Lemma to the n red chords and checking how the number of green points varies.

Take AC and BD to be any two red chords that intersect inside the circle (Assume that the four points A, B, C, D are counter-clockwisely oriented on the circle).

After the operation, the two red chords AC and BD are now replaced with two red chords AB and CD. Take EF to be an arbitrary blue chord and consider how the number of green points on chord EF varies after such operation. Based on the positions of the two points E, F, we have the following cases:

- Case 1: Both E and F are on arc $\widehat{AB}$/arc $\widehat{BC}$/arc $\widehat{CD}$/arc $\widehat{DA}$.
  We have that the number of green points on EF is unchanged after the replacement $(AC, BD) \to (AB, CD)$.
- Case 2: One of the two points (E, F) is on arc $\widehat{AB}$, while the other point is on arc $\widehat{BC}$.
  We have that the number of green points on EF is unchanged, where one green point transits from red chord DB to red chord AB.
- Case 3: One of the two points (E, F) is on arc $\widehat{AB}$, while the other point is on arc $\widehat{AD}$.
  We have that the number of green points on EF is unchanged, where one green point transits from red chord AC to red chord AB.
- Case 4: One of the two points (E, F) is on arc $\widehat{BC}$, while the other point is on arc $\widehat{CD}$.
  We have that the number of green points on EF is unchanged, where one green point transits from red chord CA to red chord CD.
- Case 5: One of the two points (E, F) is on arc $\widehat{CD}$, while the other point is on arc $\widehat{DA}$.
  We have that the number of green points on EF is unchanged, where one green point transits from red chord DB to red chord DC.
- Case 6: One of the two points (E, F) is on arc $\widehat{AB}$, while the other point is on arc $\widehat{CD}$.
  We have that the number of green points on EF is unchanged, where one green point transits from red chord AC to red chord AB and one green point transits from red chord BD to red chord CD.
- Case 7: One of the two points (E, F) is on arc $\widehat{AD}$, while the other point is on arc $\widehat{BC}$.
  We have that the number of green points on EF decrements by 2, where one green point on red chord AC and one green point on red chord BD both vanish.

Combining the 7 cases above indicates that after performing the operation described in the Lemma to two intersecting red chords once, the number of green points in the configuration is non-increasing. By the Lemma we have that such operation must terminate after finitely many steps. Consider the number of green points in the terminating configuration, where any two red chords can't intersect at any point inside the circle.

Pick an arbitrary red chord XY in the terminating configuration. Consider the arc $\widehat{XY}$ (either side works). If the number of red points on $\widehat{XY}$ is an odd number, then we have that the red chord XY must intersect with some other red chord at a point inside the circle, which does not conform to our assumption of the terminating configuration. Hence, the number of red points on arc $\widehat{XY}$ must be an even number. Given that the blue points and the red points are placed alternately on the circle, we can further deduce that the number of blue points on arc $\widehat{XY}$ must be odd. Hence, the red chord must intersect with some blue chord at some point inside the circle. Equivalently speaking, any red chord XY must contain at least one green point. From the arbitrariness of red chord XY, we can further deduce that the total number of green points in the terminating configuration must be at least $1 \cdot n = n$. To show that the minimum n can be attained, consider the following example:

Assume the $4n$ equally spaced points are $A_1, A_2, \cdots, A_{4n}$ (oriented in a counterclockwise order). We have that A_1A_{2n+1} is exactly the diameter of the circle, where A_i, A_{4n+2-i} $(2 \leq i \leq 2n)$ are symmetric with respect to the diameter. Do the following coloring:

- Points A_1, A_{2n+1} are colored blue
- Points $A_{2k}, A_{4n+2-2k}$ $(1 \leq k \leq n)$ are colored red
- Points $A_{2k+1}, A_{4n+1-2k}$ $(1 \leq k \leq n-1)$ are colored blue

Then, there are exactly n green points in the configuration, each of which lies on the diameter A_1A_{2n+1}. $\square$

5. *(8 pts)* Let n be a positive integer, and let a, b, c be real numbers.

(a) *(2 pts)* Given that $a\cos x + b\cos 2x + c\cos 3x \geq -1$ for all reals x, find, with proof, the maximum possible value of $a + b + c$.

(b) *(6 pts)* Let f be a degree n polynomial with real coefficients. Suppose that $|f(z)| \leq \left|f(z) + \frac{2}{z}\right|$ for all complex z lying on the unit circle. Find, with proof, the maximum possible value of $f(1)$.

Solution (a):

Define $h(x) = a\cos x + b\cos 2x + c\cos 3x$. Then, $h(x) \geq -1$ for all $x \in \mathbb{R}$. Plugging in $x = 0, \frac{\pi}{2}, \pi$ gives us the following three identities:

$$h(0) = a + b + c,\ h\left(\frac{\pi}{2}\right) = -b,\ h(\pi) = -a + b - c$$

Hence, $0 = h(0) + 2h(\frac{\pi}{2}) + h(\pi)$. Note that $h(x) \geq -1$ for all $x \in \mathbb{R}$, which gives us the following upper bound:

$$a + b + c = h(0) = -2h\left(\frac{\pi}{2}\right) - h(\pi) \geq 3 \Rightarrow a + b + c \leq 3.$$

Moreover, by picking $a = \frac{3}{2}, b = 1$ and $c = \frac{1}{2}$, we have that

$$\begin{aligned} h(x) &= \frac{3}{2}\cos x + \cos 2x + \frac{1}{2}\cos 3x = \frac{3}{2}\cos x + (2\cos^2 x - 1) + \frac{1}{2}(4\cos^3 x - 3\cos x) \\ &= 2\cos^3 x + 2\cos^2 x - 1 = 2\cos^2 x(\cos x + 1) - 1 \geq -1. \end{aligned}$$

Thus, the maximum possible value of $a + b + c$ is $\boxed{3}$. $\square$

Solution (b):

For any $z \in \mathbb{C}, |z| = 1$, we have that

$$\begin{aligned} |f(z)|^2 \leq \left|f(z) + \frac{2}{z}\right|^2 &\iff \overline{f(z)}f(z) \leq \left(\overline{f(z)} + \frac{2}{\overline{z}}\right)\left(f(z) + \frac{2}{z}\right) \\ \iff 2\left(zf(z) + \overline{z}\overline{f(z)}\right) + 4 \geq 0 &\iff \frac{1}{2}\left(zf(z) + \overline{zf(z)}\right) \geq -1. \end{aligned}$$

Letting $g(z) = zf(z) = a_0z + a_1z^2 + \cdots + a_nz^{n+1} = \sum_{j=0}^{n} a_j z^{j+1}$, the problem constraint can be equivalently written as $\operatorname{Re}(g(z)) \geq -1$. Moreover, we may write $z = \cos\theta + i\sin\theta$, $(\theta \in [0, 2\pi))$. Then the inequality above is

$$h(\theta) := \operatorname{Re}(g(z)) = a_0\cos(\theta) + a_1\cos(2\theta) + \cdots a_n\cos((n+1)\theta) \geq -1, \theta \in [0, 2\pi)$$

Define $\theta_k = \frac{2k\pi}{n+2}$ for $0 \leq k \leq n+1$. By substituting θ_k's into $h(\theta)$ above and summing the results, we deduce that

$$
\begin{aligned}
\sum_{k=0}^{n+1} h(\theta_k) &= \sum_{k=0}^{n+1}\sum_{j=0}^{n} a_j \cos((j+1)\theta_k) = \sum_{j=0}^{n} a_j \sum_{k=0}^{n+1} \cos((j+1)\theta_k) \\
&= \sum_{j=0}^{n} a_j \sum_{k=0}^{n+1} \mathrm{Re}(e^{i(j+1)\theta_k}) = \sum_{j=0}^{n} a_j \mathrm{Re}\left(\sum_{k=0}^{n+1} e^{i(j+1)\theta_k}\right) \\
&= \sum_{j=0}^{n} a_j \mathrm{Re}\left(\sum_{k=0}^{n+1} \left(e^{i(j+1)\theta_1}\right)^k\right) = \sum_{j=0}^{n} a_j \mathrm{Re}\left(\frac{1-e^{i(j+1)(n+2)\theta_1}}{1-e^{i(j+1)\theta_1}}\right) \\
&= \sum_{j=0}^{n} a_j \mathrm{Re}\left(\frac{0}{1-e^{i(j+1)\theta_1}}\right) = 0.
\end{aligned}
$$

This yields

$$
f(1) = \sum_{j=0}^{n} a_j = h(0) = -\sum_{k=1}^{n+1} h(\theta_k) \leq -\sum_{k=1}^{n+1}(-1) = n+1.
$$

Thus, $f(1) \leq n+1$. Now, we justify that this maximum is attainable.

Take $a_k = \frac{2(n+1-k)}{n+2}$ for $0 \leq k \leq n$. On the one hand,

$$
f(1) = \sum_{j=0}^{n} a_j = \sum_{j=0}^{n} \frac{2(n+1-j)}{n+2} = \frac{2}{n+2} \cdot \frac{(n+2)(n+1)}{2} = n+1.
$$

On the other hand, for all $\theta \in [0, 2\pi)$, we observe that

$$
\begin{aligned}
h(\theta) &= \sum_{j=0}^{n} a_j \cos((j+1)\theta) = \sum_{j=0}^{n} \frac{2(n+1-j)}{n+2} \cos((j+1)\theta) \\
&= \frac{2}{n+2}((n+1)\cos(\theta) + n\cos(2\theta) + \cdots + 2\cos(n\theta) + \cos((n+1)\theta)) \\
&= \frac{2}{n+2} \sum_{s=1}^{n+1} \left(\sum_{j=1}^{s} \cos(j\theta)\right).
\end{aligned}
$$

If $\theta = 0$, then we directly see that $h(0) = \sum_{j=0}^{n} a_j = n+1 \geq -1$. Now we only have to consider the case when $\theta \neq 0$. For any $1 \leq s \leq n+1$, we have the following identity:

$$
\begin{aligned}
\sum_{j=1}^{s} \cos(j\theta) &= \frac{1}{2\sin(\frac{\theta}{2})} \sum_{j=1}^{s} 2\cos(j\theta)\sin\left(\frac{\theta}{2}\right) = \frac{1}{2\sin(\frac{\theta}{2})} \sum_{j=1}^{s} \left(\sin\left(j\theta + \frac{\theta}{2}\right) - \sin\left(j\theta - \frac{\theta}{2}\right)\right) \\
&= \frac{1}{2\sin(\frac{\theta}{2})} \left(\sin\left(s\theta + \frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\right).
\end{aligned}
$$

Plugging in the identity above helps evaluate $h(\theta)$:

$$
\begin{aligned}
h(\theta) &= \frac{2}{n+2}\sum_{s=1}^{n+1}\frac{\sin(s\theta+\frac{\theta}{2})-\sin(\frac{\theta}{2})}{2\sin(\frac{\theta}{2})} = \frac{1}{(n+2)\sin(\frac{\theta}{2})}\left(\sum_{s=0}^{n+1}\sin\left(s\theta+\frac{\theta}{2}\right)\right)-1 \\
&= \frac{1}{(n+2)\sin^2(\frac{\theta}{2})}\left(\sum_{s=0}^{n+1}\sin\left(s\theta+\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)-1 \\
&= \frac{1}{(n+2)\sin^2(\frac{\theta}{2})}\left(\sum_{s=0}^{n+1}\cos(s\theta)-\cos((s+1)\theta)\right)-1 \\
&= \frac{1-\cos((n+2)\theta)}{(n+2)\sin^2(\frac{\theta}{2})}-1 \geq -1.
\end{aligned}
$$

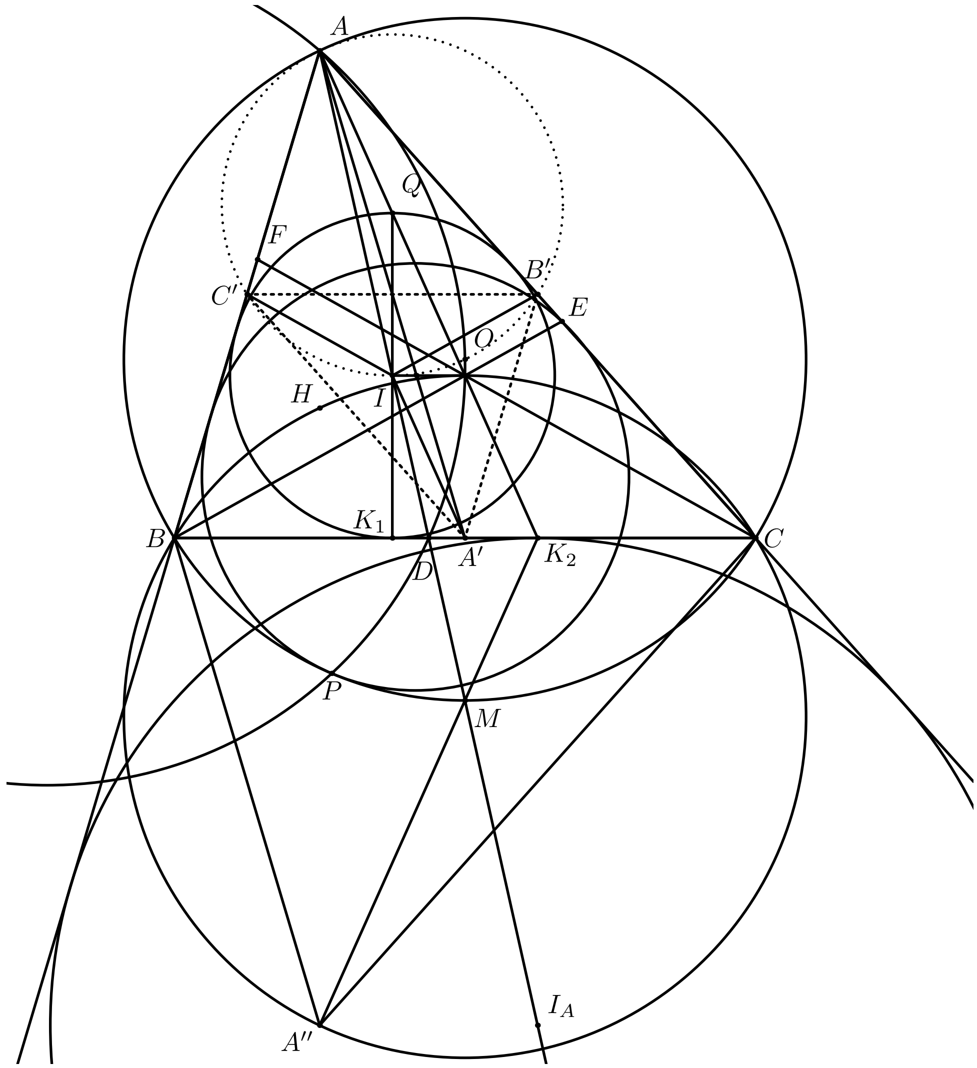
This proves that $f(1)=n+1$ can be attained, so the maximum possible value of $f(1)$ is $\boxed{n+1}$. □

6. *(9 pts)* Let ABC be a triangle with circumcenter O. The interior bisector of $\angle BAC$ intersects BC at D. Circle ω_A is tangent to segments AB and AC and internally tangent to the circumcircle of ABC at the point P. Let E and F be the respective points at which the B-excircle and C-excircle of ABC are tangent to AC and AB. Suppose that lines BE and CF pass through a common point N on the circumcircle of AEF.

Note: for a triangle ABC, the A-excircle is the circle lying outside triangle ABC that is tangent to side BC and the extensions of sides AB, AC. The B, C-excircles are defined similarly.

(a) *(7 pts)* Prove that the circumcircle of PDO passes through N.

(b) *(2 pts)* Suppose that $\frac{PD}{BC}=\frac{2}{7}$. Find, with proof, the value of $\cos(\angle BAC)$.



Solution (a):

Note that N is the Nagel point of $\triangle ABC$. Let H be the orthocenter of $\triangle ABC$. Since $N \in (AEF)$, $m\angle ENF = 180^\circ - m\angle BAC = m\angle BHC$, so N lies on (BHC) (i.e., the circumcircle of $\triangle BHC$).

Let $\triangle A'B'C'$ be the medial triangle of $\triangle ABC$ (with A' on BC, B' on AC, and C' on AB), G be the centroid of ABC, and I be the incenter of ABC. We first prove a lemma.

Lemma: I is the Nagel Point of $\triangle A'B'C'$ (the line IN is called the *Nagel line*).

Proof of lemma: Let K_1 and K_2 be the respective points at which the incircle and excircle of ABC touches BC, and let Q be the antipode of K_1 on the incircle. The incircle and excircle are homothetic with respect to A, so A, Q, N, K_2 are collinear. Since I and A' are the respective midpoints of K_1Q and K_1K_2, we have that $IA' \parallel AK_2$. However, $A'C' \parallel AC$ and $A'B' \parallel AB$, so $\angle C'A'I \cong \angle CAN, \angle B'A'I \cong \angle BAN$. Hence, the cevian AI in $\triangle A'B'C'$ corresponds to the cevian AN in $\triangle ABC$. Likewise, the cevians BI and CI in $\triangle A'B'C'$ correspond to respective cevians BN and CN in $\triangle ABC$. So I is the Nagel Point of $\triangle A'B'C'$ as desired.

Thus, consider the homothety $\mathscr{H}$ with center G and factor $-\frac{1}{2}$. Under $\mathscr{H}$,

$$\begin{aligned}\triangle ABC &\longmapsto \triangle A'B'C'\\ H &\longmapsto O \text{ by the Euler Line}\\ N &\longmapsto I \text{ by the above lemma and correspondence of similarity}\end{aligned}$$

In particular, B', C', O, I are concyclic. However, $B'O$ and $C'O$ are perpendicular bisectors of sides AC and AB, so $AB'OC'$ is cyclic. Hence, $I \in (ABC)$. Since AI bisects $\angle BAC$, we see that $\angle C'B'I \cong \angle B'C'I \implies B'I = C'I$. By correspondence of similarity, $BN = CN$. Thus, N lies on the midpoint of $\widehat{BHC}$ of (BHC).

Reflect A, N, K_2 over BC to points A'', M, K_2. Since (BHC) and (ABC) are congruent and symmetric about BC, M is the midpoint of $\widehat{BC}$ on (ABC). Let $\mathscr{J}$ denote the composition of the inversion with center A and radius $\sqrt{AB \cdot AC}$ followed by reflection over AD. It is well known under $\mathscr{J}$,

$$\begin{aligned}M &\longmapsto D\\ A'' &\longmapsto O\\ \text{The } A\text{-excircle} &\longmapsto \omega_A \implies K_2 \longmapsto P.\end{aligned}$$

In particular, since A'', M, K_2 are collinear, we see that A, P, D, O are concyclic.

Finally, observe that $MD = ND$ and $MO = AO$. Since M lies on MO, the perpendicular bisector of BC, we see that $\triangle AOM \sim \triangle NDM$ (they are both isosceles and share the common $\angle AMO$). Thus, ND and AO are antiparallel with respect to M, so A, O, N, D are concyclic. Thus, P, D, O, N are concyclic as desired. $\square$

Solution (b):

We use some results from our solution to (a).

Since I lies on $(AB'C')$, we have that I bisects AM. Furthermore, by Fact 5, $IM = MC$, so $2MC = AM$. Since $\angle ABC \cong \angle AMC$ and AM bisects $\angle BAC$, we see that $\triangle ABD \sim \triangle AMC$. Thus, by the Angle Bisector Theorem, $\frac{AC}{DC} = \frac{AB}{DB} = 2$. Furthermore, since $\frac{CM}{MD} = \frac{AB}{DB} = 2$, we observe that D bisects IM and $b + c = 2a$ (where we use standard conventions for triangle side lengths). Now, cyclic quadrilateral $APDN$ and $\angle PAD \cong \angle DAN$ yields $\angle DPN \cong \angle DNP \implies PD = DN$. However, $ND = MD$, so $2PD = AI$. The given condition is therefore $\frac{AI}{BC} = \frac{4}{7}$.

Let $\alpha = \angle BAC$, r be the inradius of ABC, and R be the circumradius of ABC. We first claim that $r = R(1 - \cos\alpha)$. If we take E' to be the tangency point of the incircle of ABC and side AB, then we have that $s - a = AE' = \frac{r}{\tan\frac{\alpha}{2}} = \frac{r(1+\cos\alpha)}{\sin\alpha}$. However, since $b + c = 2a$, $BC = 2 \cdot AE'$. By the Law of Sines, $BC = 2R\sin\alpha$ as well. Thus, $R\sin\alpha = \frac{r(1+\cos\alpha)}{\sin\alpha} \implies r = R(1 - \cos\alpha)$.

By the Law of Sines and trigonometry, we have that

$$\frac{AI}{BC} = \frac{\frac{r}{\sin\frac{\alpha}{2}}}{2R\sin\alpha} = \frac{1 - \cos\alpha}{2\sin\alpha\sin\frac{\alpha}{2}} = \frac{\tan\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}} = \frac{1}{2\cos\frac{\alpha}{2}} = \frac{4}{7}.$$

Hence, $\cos\frac{\alpha}{2} = \frac{7}{8} \implies \sin\frac{\alpha}{2} = \frac{\sqrt{15}}{8}$. By the double angle formula, $\cos(\angle BAC) = \cos\alpha = \boxed{\frac{17}{32}}$.