CHMMC 2021-2022

Individual Round Solutions

Problem 1. Fleming has a list of 8 mutually distinct integers between 90 to 99, inclusive. Suppose that the list has median 94, and that it contains an even number of odd integers. If Fleming reads the numbers in the list from smallest to largest, then determine the sixth number he reads.

Solution: $\boxed{96}$.
Denote the integers $x_1 < x_2 < \cdots < x_8$. Since the median is 94,

$$(x_4, x_5) \in \{(93,95),(92,96),(91,97),(90,98)\}$$

However, the last 3 cases cannot happen, since there are 8 distinct integers. Thus, the first five numbers in increasing order are $90,91,92,93,95$. If the sixth number is 97, then the numbers are $90,91,92,93,95,97,98,99$ which fails—it contains an odd number of odd integers. Hence, as there are no other choices besides 96, the sixth number must be $\boxed{96}$.

Problem 2. Find the number of ordered pairs (x,y) of three digit base-10 positive integers such that $x-y$ is a positive integer, and there are no borrows in the subtraction $x-y$. *For example, the subtraction on the left has a borrow at the tens digit but not at the units digit, whereas the subtraction on the right has no borrows.*

$$\begin{array}{cccc} & 4 & 7 & 2 \\ - & 1 & 9 & 1 \\ \hline & 2 & 8 & 1 \end{array} \qquad \begin{array}{cccc} & 3 & 7 & 9 \\ - & 2 & 6 & 3 \\ \hline & 1 & 1 & 6 \end{array}$$

Solution: $\boxed{135225}$.
We claim that no borrows occur iff every digit of x is greater or equal to every digit of y. If this property holds, it is clear that no borrows occur. If this property does not hold, consider the first place where the corresponding digit of x is less than the corresponding digit of y. By minimality, a borrow occurs at that place.
For the leftmost digit, there are $\binom{9}{2}+9=\binom{10}{2}$ choices, whereas there are $\binom{10}{2}+10=\binom{11}{2}$ for the remaining digits since x,y are both three digits. But for each possible x, we subtract 1 from the total number of ordered pairs, since $x-y>0$. There are 900 three-digit numbers, so the answer is $\binom{10}{2}\binom{11}{2}^2-900=\boxed{135225}$.

Problem 3. Evaluate

$$1\cdot 2\cdot 3-2\cdot 3\cdot 4+3\cdot 4\cdot 5-4\cdot 5\cdot 6+\cdots+2017\cdot 2018\cdot 2019-2018\cdot 2019\cdot 2020+1010\cdot 2019\cdot 2021.$$

Solution: $\boxed{3060300}$.
Put $n=1009$, and let S be the desired value. Grouping consecutive terms in the above expression—$k(k+1)(k+2)-(k+1)(k+2)(k+3)=-3(k+1)(k+2)$—notice that

$$\begin{aligned} S &= -3(2\cdot 3+4\cdot 5+\cdots+2018\cdot 2019)+\frac{1}{2}(2n+1)(2n+2)(2n+3) \\ &= -3\left(\sum_{i=1}^{n} 2i(2i+1)\right)+(n+1)(2n+1)(2n+3) \\ &= -6\left(2\cdot\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right)+(n+1)(2n+1)(2n+3); \end{aligned}$$

we use "sum of first n square numbers" formula. The last expression simplifies to $3(n+1)^2$, so the answer is $\boxed{3060300}$.

Problem 4. Find the number of ordered pairs of integers (a,b) such that

$$\frac{ab+a+b}{a^2+b^2+1}$$

is an integer.

Solution: $\boxed{3}$.

Observe that

$$(a-b)^2+(a-1)^2+(b-1)^2 \geq 0 \implies ab+a+b \leq a^2+b^2+1 \implies \frac{ab+a+b}{a^2+b^2+1} \leq 1$$
$$(a+b)^2+(a+1)^2+(b+1)^2 \geq 0 \implies -ab-a-b \leq a^2+b^2+1 \implies \frac{ab+a+b}{a^2+b^2+1} \geq -1$$

Hence, $\frac{ab+a+b}{a^2+b^2+1} \in \{0, \pm 1\}$.

If $\frac{ab+a+b}{a^2+b^2+1} = 1$, then $(a-b)^2+(a-1)^2+(b-1)^2 = 0$ which has one real solution $(a,b) = (1,1)$.

If $\frac{ab+a+b}{a^2+b^2+1} = -1$, then $(a+b)^2+(a+1)^2+(b+1)^2 = 0$ which has no real solutions.

If $\frac{ab+a+b}{a^2+b^2+1} = 0$, then $ab+a+b = 0 \implies (a+1)(b+1) = 1$. This diophantine equation has two solutions $(a,b) = (0,0), (a,b) = (-2,-2)$.

Thus, we have $\boxed{3}$ total ordered pairs.

Problem 5. Lin Lin has a 4×4 chessboard in which every square is initially empty. Every minute, she chooses a random square C on the chessboard, and places a pawn in C if it is empty. Then, regardless of whether C was previously empty or not, she then immediately places pawns in all empty squares a king's move away from C. The expected number of minutes before the entire chessboard is occupied with pawns equals $\frac{m}{n}$ for relatively prime positive integers m,n. Find $m+n$.

A king's move, in chess, is one square in any direction on the chessboard: horizontally, vertically, or diagonally.

Solution: $\boxed{28}$.

Subdivide the chessboard into four 2×2 quadrants Q_1,Q_2,Q_3,Q_4. Let $C \in Q_i$ be a corner square of the chessboard. At any given time, there is a pawn on C iff Lin Lin has chosen a square in Q_i at some point in time. Hence, it is necessary and sufficient for Lin Lin to have chosen at least one square from each of Q_1,Q_2,Q_3,Q_4 for the chessboard to be filled with pawns.

Let $f(n)$ denote the expected number of minutes until Lin Lin has chosen n quadrants. With n quadrants chosen, the next square Lin Lin chooses corresponds to a new quadrant with probability $\frac{4-n}{4}$. This yields the recursion $f(n+1) = \frac{4-n}{4}f(n) + \frac{n}{4}f(n+1)+1$, for $n = 0,1,2,3$. Clearly $f(0) = 0$, so solving this linear system yields $f(4) = 4(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}) = \frac{25}{3}$, so the answer is $\boxed{28}$.

Problem 6. Let $P(x) = x^5-3x^4+2x^3-6x^2+7x+3$ and $\alpha_1,\ldots,\alpha_5$ be the roots of $P(x)$. Compute

$$\prod_{k=1}^{5}(\alpha_k^3-4\alpha_k^2+\alpha_k+6).$$

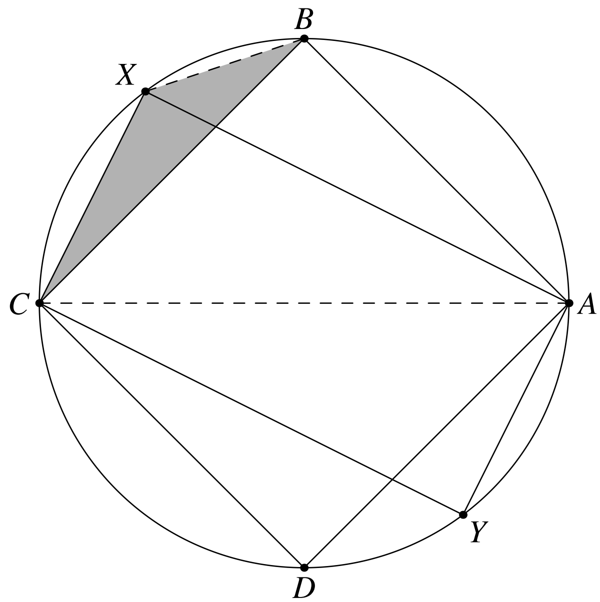
Solution: $\boxed{-2688}$.

Factoring $P(x) = \prod_{k=1}^{5}(x-\alpha_k), x^3-4x^2+x+6 = (x-2)(x+1)(x-3)$, the desired value comes out to be

$$\prod_{k=1}^{5}(\alpha_k-2)(\alpha_k+1)(\alpha_k-3) = (-1)^{15}\prod_{k=1}^{5}(2-\alpha_k)(-1-\alpha_k)(3-\alpha_k) = -P(2)P(-1)P(3) = \boxed{-2688}.$$

Problem 7. Rectangle $AXCY$ with a longer length of 11 and square $ABCD$ share the same diagonal $\overline{AC}$. Assume B, X lie on the same side of $\overline{AC}$ such that triangle BXC and square $ABCD$ are non-overlapping. The maximum area of BXC across all such configurations equals $\frac{m}{n}$ for relatively prime positive integers m, n. Compute $m+n$.

Solution: $\boxed{137}$.



Hexagon $ABXCDY$ is clearly "convex" and cyclic. Set $AB = BC = a$, so $AC = a\sqrt{2}$. We can apply Ptolemy on $ABXC$:

$$a \cdot XC + a\sqrt{2} \cdot XB = 11a \implies XC + XB\sqrt{2} = 11.$$

By AM-GM inequality, $XC + XB\sqrt{2} \geq 2\sqrt{XC \cdot XB\sqrt{2}}$, so $\frac{121\sqrt{2}}{8} \geq XC \cdot XB$.

Noting $\angle CXB = 180^\circ - \angle BAC = 135^\circ$, we compute $[BXC] = \frac{1}{2} \cdot XB \cdot XC \cdot \sin(\angle CXB) \leq \frac{121}{16}$ and equality is achievable when $XC = \frac{11}{2}$. The answer is $\boxed{137}$.

Problem 8. Earl the electron is currently at $(0,0)$ on the Cartesian plane and trying to reach his house at point $(4,4)$. Each second, he can do one of three actions: move one unit to the right, move one unit up, or teleport to the point that is the reflection of its current position across the line $y = x$. Earl cannot teleport in two consecutive seconds, and he stops taking actions once he reaches his house.

Earl visits a chronologically ordered sequence of distinct points $(0,0), \ldots, (4,4)$ due to his choice of actions. This is called an *Earl-path*. How many possible such Earl-paths are there?

Solution: $\boxed{3584}$.

Consider the continuous (i.e., no teleporting) Earl-paths from $(0,0)$ to $(4,4)$ without going over the line $y = x$. This is simply the 4th Catalan number $\frac{1}{5}\binom{8}{4}$. We call such path a *Catalan path*.

Any (possibly discontinuous) Earl-path $\mathscr{P}$ may be considered as a union of 8 unit line segments. Consider the following transformation: for every segment in $\mathscr{P}$ over $y = x$, reflect it across $y = x$. The image of $\mathscr{P}$ under this transformation is necessarily an Earl-path that does not go over $y = x$, hence a Catalan path. Thus, it suffices to choose which of the 8 segments are above the line $y = x$, with 2 choices for each segment, over the collection of all Catalan paths.

Thus, the final answer is $\frac{2^8}{5} \cdot \binom{8}{4} = \boxed{3584}$.

Problem 9. Let $P(x)$ be a degree-2022 polynomial with leading coefficient 1 and roots $\cos(\frac{2\pi k}{2023})$ for $k = 1, \ldots, 2022$ (note $P(x)$ may have repeated roots). If $P(1) = \frac{m}{n}$ where m and n are relatively prime positive integers, then find the remainder when $m + n$ is divided by 100.

Solution: $\boxed{33}$.

The task is to compute

$$\prod_{k=1}^{2022} \left(1 - \cos\left(\frac{2\pi k}{2023}\right)\right).$$

Let $A_0 A_1 \ldots A_{2022}$ be a regular 2023-gon inscribed in a unit circle centered at O. By the Law of Cosines on $\triangle A_0 O A_k$,

$$1 - \cos\left(\frac{2\pi k}{2023}\right) = \frac{A_0 A_k^2}{2}.$$

Put $\zeta = e^{\frac{2\pi i}{2023}} \in \mathbb{C}$; we have $A_0 A_k = |1 - \zeta^k|$, so the desired product is

$$\frac{1}{2^{2022}} \left(\prod_{k=1}^{2022} A_0 A_k\right)^2 = \frac{1}{2^{2022}} \left(\prod_{k=1}^{2022} |1 - \zeta^k|\right)^2 = \frac{1}{2^{2022}} \left|\prod_{k=1}^{2022} (1 - \zeta^k)\right|^2.$$

The polynomial with roots $\zeta, \ldots, \zeta_{2022}$ is $f(x) = x^{2022} + \cdots + x + 1$, so

$$2023 = |f(1)| = \left|\prod_{k=1}^{2022} (1 - \zeta^k)\right| \implies \frac{1}{2^{2022}} \left|\prod_{k=1}^{2022} (1 - \zeta^k)\right|^2 = \frac{2023^2}{2^{2022}}.$$

Note $2023^2 \equiv 29 \pmod{100}$ and by Euler's Theorem $2^{2022} \equiv 2^2 \equiv 4 \pmod{25}$. Then, $2^{2022} \equiv 4 \pmod{100}$ and the answer is $\boxed{33}$.

Problem 10. A randomly shuffled standard deck of cards has 52 cards, 13 of each of the four suits. There are 4 Aces and 4 Kings, one of each of the four suits. One repeatedly draws cards from the deck until one draws an Ace. Given that the first King appears before the first Ace, what is the expected number of cards one draws after the first King and before the first Ace?

Solution: $\boxed{424}$.

The idea is that the Aces and Kings form 8 dividers which partition the remaining 44 cards into 9 classes. The expected number of cards per class equals $\frac{44}{9}$. Then, supposing that there are $k \geq 1$ Kings before the first Ace, the expected number of cards drawn after the first King and before stopping is $\frac{44k}{9} + k - 1$.

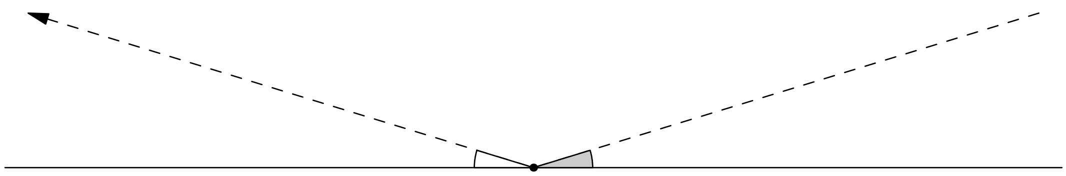
Let C be the number of cards drawn between the first King and the first Ace, K be the number of Kings before the first Ace, and A the event that the first King appears before the first Ace. The number of permutations of 4

Kings and 4 Aces with k Kings before the first Ace is $\binom{7-k}{3}$. Hence, $\mathbb{P}(K=k) = \frac{\binom{7-k}{3}}{\binom{8}{4}}$, so

$$\begin{aligned}
\mathbb{E}[C|A] = \sum_{k=1}^{4} \mathbb{P}(K=k|A)\mathbb{E}(C|K=k,A) &= \sum_{k=1}^{4} \frac{\mathbb{P}(K=k)}{\frac{1}{2}} \cdot \left(\frac{44k}{9} + k - 1\right) \\
&= -1 + \frac{106}{9} \cdot \sum_{k=1}^{4} \frac{k\binom{7-k}{3}}{\binom{8}{4}} \\
\text{(apply hockey stick identity)} &= -1 + \frac{106}{9} \cdot \frac{\binom{8}{5}}{\binom{8}{4}} \\
&= \frac{379}{45}.
\end{aligned}$$

The answer is $\boxed{424}$.

Problem 11. The following picture shows a beam of light (dashed line) reflecting off a mirror (solid line). The *angle of incidence* is marked by the shaded angle; the *angle of reflection* is marked by the unshaded angle.



The sides of a unit square $ABCD$ are magically distorted mirrors such that whenever a light beam hits any of the mirrors, the measure of the angle of incidence between the light beam and the mirror is a positive real constant θ degrees greater than the measure of the angle of reflection between the light beam and the mirror. A light beam emanating from A strikes $\overline{CD}$ at W_1 such that $2DW_1 = CW_1$, reflects off of $\overline{CD}$ and then strikes $\overline{BC}$ at W_2 such that $2CW_2 = BW_2$, reflects off of $\overline{BC}$, etc. To this end, denote W_i the ith point at which the light beam strikes $ABCD$.

As i grows large, the area of $W_iW_{i+1}W_{i+2}W_{i+3}$ approaches $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

Solution: $\boxed{39}$

Let $A = W_0$. The difference between the angle of incidence and the angle of reflection θ is clearly $\angle AW_1D - \angle W_2W_1C = \tan^{-1}(3) - \tan^{-1}(\frac{1}{2})$. However, $\tan^{-1}(3) - \tan^{-1}(\frac{1}{2}) = \tan^{-1}(2) - \tan^{-1}(\frac{1}{3}) = \angle W_1W_2C - \angle W_3W_2B$. Since $\angle W_1W_2C = \tan^{-1}(2)$, we see that $\angle W_3W_2B = \tan^{-1}(\frac{1}{3}), \angle W_2W_3B = \tan^{-1}(3)$. This suggests a pattern of similar right triangles. By drawing the path $W_0 \to W_1 \to W_2 \to W_3 \to \dots$ and repeatedly chasing angles, we observe that

$$\frac{DW_{4i}}{W_{4i+1}D} = \frac{BW_{4i+2}}{W_{4i+3}B} = 3$$
$$\frac{CW_{4i+1}}{W_{4i+2}C} = \frac{AW_{4i+3}}{W_{4i+4}A} = 2$$

for all nonnegative integers i. In other words, we have the following two equivalence classes of similar right triangles:

$$\{\triangle W_{4i}DW_{4i+1}, \triangle W_{4i+2}BW_{4i+3}\}_{i\geq 0}, \{\triangle W_{4i+1}CW_{4i+2}, \triangle W_{4i+3}AW_{4i+4}\}_{i\geq 0}.$$

Define the sequence:

$$b_1, b_2, b_3, b_4, \dots = CW_2, AW_4, CW_6, AW_8, \dots$$

The above ratios, along with the fact $ABCD$ is a unit square, allow us to compute any b_i. In particular, a general length chasing process yields the recursive formula $b_{i+1} = \frac{2+b_i}{6}$. An induction argument implies that $b_i = \frac{6^i-1}{3\cdot 6^{i-1}\cdot 5}$.

For $j = 0,1,2,3$, denote $W_{\infty,0} = \lim_{i\to\infty} W_{4i+j}$. The formula for b_i implies $AW_{\infty,0} = CW_{\infty,2} = \frac{2}{5}$. Similarly, we may show $DW_{\infty,1} = BW_{\infty,1} = \frac{1}{5}$. By continuity of area of a quadrilateral,

$$\begin{aligned}[W_{\infty,0}W_{\infty,1}W_{\infty,2}W_{\infty,3}] &= 1 - [W_{\infty,1}DW_{\infty,0}] - [W_{\infty,2}CW_{\infty,1}] - [W_{\infty,3}BW_{\infty,2}] - [W_{\infty,0}AW_{\infty,3}] \\ &= 1 - \frac{1}{2}\cdot\left(\frac{3}{5}\cdot\frac{1}{5} + \frac{4}{5}\cdot\frac{2}{5} + \frac{3}{5}\cdot\frac{1}{5} + \frac{4}{5}\cdot\frac{2}{5}\right) = \frac{14}{25}\end{aligned}$$

and the answer is $\boxed{39}$.

Problem 12. For any positive integer m, define $\varphi(m)$ the number of positive integers $k \le m$ such that k and m are relatively prime. Find the smallest positive integer N such that $\sqrt{\varphi(n)} \ge 22$ for any integer $n \ge N$.

Solution: $\boxed{2311}$.

For arbitrary $n \in \mathbb{N}$, we write its prime factorization $n = q_1^{\beta_1} q_2^{\beta_2} \dots q_t^{\beta_t}$ throughout this solution, where the primes $q_1, \dots, q_t$ are in increasing order.

Lemma 1: For any integer $n > 2$, $\varphi(n) \ne 481, 482, 483$.

Proof 1: First note $\gcd(a,n) = 1 \iff \gcd(n-a,n) = 1$ for all integers $1 \le a < n$, implying $\varphi(n) \equiv 0 \pmod 2$ if $\varphi(n) > 1$, so $\varphi(n) \ne 481, 483$.

Now assume on the contrary there is some integer n such that $\varphi(n) = 482 = 2\cdot 241$ (note 241 is prime). Since

$$\varphi(n) = \prod_{j=1}^{t} q_j^{\beta_j - 1}(q_j - 1),$$

241 necessarily divides one of $q_j, q_j - 1$. If $241 \mid q_j$, then $241 = q_j$, so $241^2 \mid n \implies \varphi(n) \ge q_j(q_j-1) = 241\cdot 240$, a contradiction. If $241 \mid q_j - 1$, then $q_j = 1 + 241k$ for some positive integer k. However, $q_j \ne 242, 483$ (for $k = 1,2$) as those numbers are composite; thus, $\varphi(n) \ge q_j - 1 \ge 241\cdot 3 > 482$, a contradiction. Hence, $\varphi(n) \ne 482$. ■

Let $p_1 = 2, p_2 = 3, p_3 = 5, \dots$ be the prime numbers listed in increasing order, and define $x_s = p_1 p_2 \dots p_s$.

Lemma 2: For all integers $n \ge x_s$, we have $\varphi(n) \ge \varphi(x_s)$ with equality iff $n = x_s$.

Proof 2: On the one hand, suppose $\frac{\varphi(n)}{n} > \frac{\varphi(x_s)}{x_s}$. By multiplying these inequalities, we obtain $\varphi(n) > \varphi(x_s)$, as needed.

On the other hand, suppose

$$\frac{(q_1-1)(q_2-1)\dots(q_t-1)}{q_1 q_2 \dots q_t} = \frac{\varphi(n)}{n} \le \frac{\varphi(x_s)}{x_s} = \frac{(p_1-1)(p_2-1)\dots(p_s-1)}{p_1 p_2 \dots p_s}.$$

Note $q_i \ge p_i \implies 1 \ge \frac{q_i-1}{q_i} \ge \frac{p_i-1}{p_i}$; hence, $t \ge s$ necessarily. Thus,

$$\varphi(n) \ge (q_1-1)\dots(q_t-1) \ge (q_1-1)\dots(q_s-1) \ge (p_1-1)\dots(p_s-1) = \varphi(x_s)$$

with equality iff $t = s$ and $q_i = p_i$ for all $1 \le i \le s$. ■

Finally, note $\varphi(x_5) = \varphi(2\cdot 3\cdot 5\cdot 7\cdot 11) = \varphi(2310) = 480$. The above two lemmas together imply $\varphi(n) \ge 484$ for all integers $n > 2310$, so the answer is $\boxed{2311}$.

Problem 13. Let n be a fixed positive integer, and let $\{a_k\}$ and $\{b_k\}$ be sequences defined recursively by

$$\begin{aligned} a_1 &= b_1 = n^{-1} \\ a_j &= j(n-j+1)a_{j-1},\ j > 1 \\ b_j &= nj^2 b_{j-1} + a_j,\ j > 1 \end{aligned}$$

When $n = 2021$, then $a_{2021} + b_{2021} = m \cdot 2017^2$ for some positive integer m. Find the remainder when m is divided by 2017.

Solution: $\boxed{1043}$.

By induction on j, it is not hard to deduce the following formulas for a_j, b_j:

$$\begin{aligned} a_j &= \frac{((n-1)!)^2 j!}{n!(n-j)!} \\ b_j &= \sum_{k=1}^{j} \frac{(j!)^2}{(k!)^2} n^{j-k} a_k. \end{aligned}$$

In the cases of a_n, b_n, the above formulas simplify to the following

$$\begin{aligned} a_n &= ((n-1)!)^2 \\ b_n &= ((n-1)!)^2 \sum_{k=1}^{n} \binom{n}{k} n^{n-k} = ((n-1)!)^2((n+1)^n - n^n), \end{aligned}$$

as $(n+1)^n - n^n = \sum_{k=1}^{n} \binom{n}{k} n^{n-k}$ by binomial expansion. Considering the fact $n = 2021$, we have $a_{2021} + b_{2021} = (2020!)^2(1 + 2022^{2021} - 2021^{2021})$. Hence,

$$\begin{aligned} & m = (2016!)^2(2018 \cdot 2019 \cdot 2020)^2 \cdot (1 + 2022^{2021} - 2021^{2021}) \\ \implies & m \equiv (-1)^2(1 \cdot 2 \cdot 3)^2(1 + 5^5 - 4^5) \equiv \boxed{1043} \pmod{2017} \end{aligned}$$

by applying Wilson's and Fermat's theorems to the prime $p = 2017$.

Problem 14. Consider the quadratic polynomial $g(x) = x^2 + x + 1020100$. A positive odd integer n is called g-friendly if and only if there exists an integer m such that n divides $2 \cdot g(m) + 2021$. Find the number of g-friendly positive odd integers less than 100.

Solution: $\boxed{18}$.

Clearly $n = 1$ is a solution, so now suppose $3 \le n < 100$. Since n is odd, $n \mid 2g(m) + 2021 \iff n \mid 2(2g(m) + 2021)$. Note $1020100 = 1010^2$, so

$$\begin{aligned} 2(2g(m) + 2021) &= 2(2m^2 + 2m + 2 \cdot 1010^2 + 2021) \\ &= 4m^2 + 4m + 4 \cdot 1010^2 + 4 \cdot 1010 + 2 \\ &= 4m^2 + 4m + (2 \cdot 1010 + 1)^2 + 1 \\ &= (2m+1)^2 + 2021^2 \end{aligned}$$

Suppose that $\gcd(n, 2021) > 1$. Noting $2021 = 43 \cdot 47$, we have $n = 43, 47$ and $n \mid 2021$. We may take $m = 21$ or 23 respectively, so $n = 43, 47$ are both g-friendly.

Otherwise, this means for all primes $p \mid n$,

$$1 = \left(\frac{-2021^2}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{2021}{p}\right)^2 = \left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}.$$

Therefore, $p \equiv 1 \pmod 4$, and this must hold for all $p \mid n$.

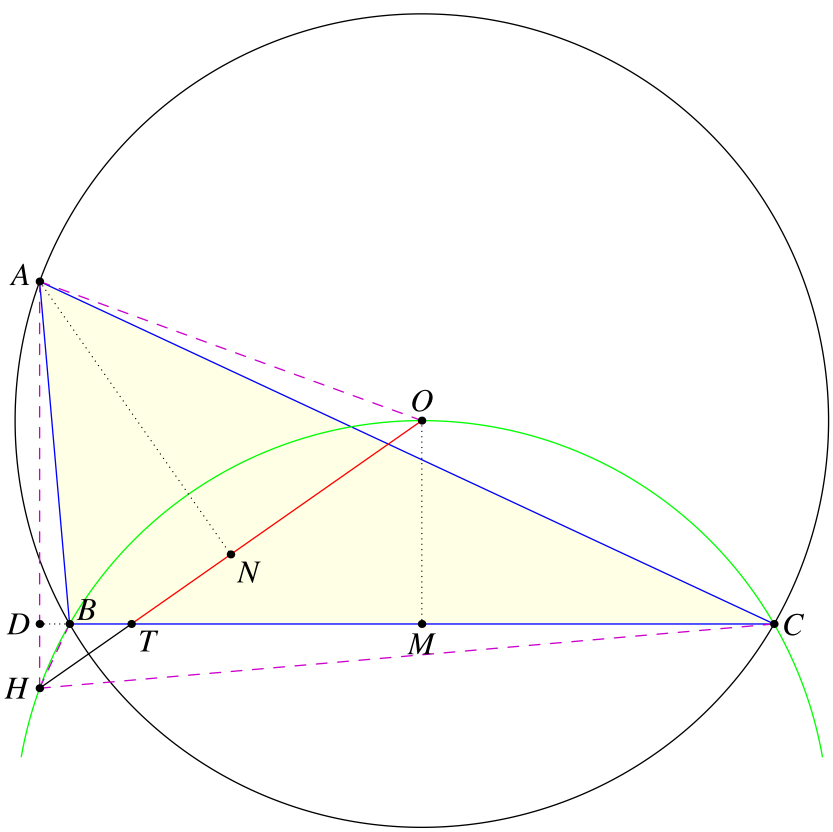
If n is a product of distinct $1 \pmod 4$ primes—perhaps a single prime—then it follows by Chinese Remainder Theorem that n is g-friendly. We can enumerate the cases—if n is a single prime, then it must be $5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 93, 97$, and if n is composite, then it must be $5 \cdot 13, 5 \cdot 17$. So there are 14 cases here.

Otherwise, if $\nu_p(n) \geq 2$ for some prime p, noting $p \equiv 1 \pmod 4$ is still needed and $n < 100$ it follows that $n = 25$. Note we can take $m = 1$, and then we have $(2 \cdot 1 + 1)^2 + 2021^2 = 3^2 + 2021^2 \equiv 9 + 41 \equiv 0 \pmod{25}$. Hence, 25 is g-friendly.

Thus, there are $1 + 2 + 14 + 1 = \boxed{18}$ g-friendly positive odd integers less than 100.

Problem 15. Let ABC be a triangle with $AB < AC$, inscribed in a circle with radius 1 and center O. Let H be the intersection of the altitudes of ABC. Let lines $\overline{OH}, \overline{BC}$ intersect at T. Suppose there is a circle passing through B, H, O, C. Given $\cos(\angle ABC - \angle BCA) = \frac{11}{32}$, then $TO = \frac{m\sqrt{p}}{n}$ for relatively prime positive integers m, n and squarefree positive integer p. Find $m + n + p$.

Solution: $\boxed{46}$.



Note $BHOC$ cyclic means $2A = \measuredangle BOC = \measuredangle BHC = 180^\circ - A$, so $A = 60^\circ$. The system of equations

$$\cos(B - C) = \cos B \cos C + \sin B \sin C = \frac{11}{32}$$
$$\cos(B + C) = \cos B \cos C - \sin B \sin C = \cos(120^\circ) = -\frac{1}{2}$$

implies $\cos B \cos C = -\frac{5}{64}$, so $\triangle ABC$ has an obtuse angle at B. Thus, H and O necessarily lie on opposite sides of $\overline{BC}$.

By angle chasing, $\measuredangle HAB = \measuredangle CAO = B - 90^\circ$. Hence, $\measuredangle HAO = 2B - 120^\circ = B - C$. Note $AH = 2R\cos A = 1 = AO$. Let N be the midpoint of $\overline{HO}$, so

$$OH = 2HN = 2AH\sin\left(\frac{B-C}{2}\right) = 2\sqrt{\frac{1-\cos(B-C)}{2}} = \frac{\sqrt{21}}{4}.$$

OTOH, let $D = \overline{AH} \cap \overline{BC}$. Then, $\measuredangle BHD = C$, so

$$HD = BH\cos(\measuredangle BHD) = 2R\cos(180^\circ - B)\cos C = \frac{5}{32}.$$

To finish, let M be the midpoint of $\overline{BC}$, and so $OM = R\sin A = \frac{1}{2}$. Then, we may apply proportionality with respect to the similarity $\triangle THD \sim \triangle TOM$, which yields $TO = \frac{\sqrt{21}}{4} \cdot \frac{\frac{1}{2}}{\frac{1}{2}+\frac{5}{32}} = \frac{4\sqrt{21}}{21}$. The answer is $\boxed{46}$.