CHMMC 2021-2022

Proof Round Solutions

Problem 1. *[4]* Find all ordered triples (a,b,c) of real numbers such that

$$(a-b)(b-c)+(b-c)(c-a)+(c-a)(a-b)=0.$$

Solution:
By factoring, we obtain $(a-b)(b-c)=(c-a)^2$ and the same cyclic expressions. Thus, $(c-a)^3=(a-b)^3=(b-c)^3=(a-b)(b-c)(c-a)$. Since a,b,c are reals, we have that $a-b=b-c=c-a$. If this common value were positive, then $a>b, b>c, c>a$, a contradiction. Likewise, this common value cannot be negative. Thus, $a=b=c$, so all such triples (a,b,c) necessarily take the form $(t,t,t), t\in\mathbb{R}$, and it is clear that all of these triples work. $\square$

Problem 2. *[4]* For any positive integer n, let $p(n)$ be the product of its digits in base-10 representation. Find the maximum possible value of $\frac{p(n)}{n}$ over all integers $n \geq 10$.

Solution: $\boxed{\dfrac{9}{11}}$.

Let S_d be the set of all (base-10) d-digit positive integers. For $n = \underline{a_d a_{d-1} \ldots a_1} \in S_d$, we claim $\frac{p(n)}{n}$ is maximized, over all such n by choosing $n = \underline{99\ldots9}$. To this end, suppose $n \in S_d$, all its digits are nonzero (as any number with 0 as a digit clearly does not maximize $\frac{p(n)}{n}$), and at place t the digit of n is $a_t < 9$. Then, by replacing a_t with 9, $p(n)$ increases by a factor of $\frac{9}{a_t}$, whereas n increases by the factor

$$\frac{\underline{a_d a_{d-1} \ldots 9 \ldots a_1}}{\underline{a_d a_{d-1} \ldots a_t \ldots a_1}} < \frac{9}{a_t}$$

This inequality may be proven by cross-multiplying and noting $a_t < 9$. Hence, such n does not maximize $\frac{p(n)}{n}$. Now, over $S_2, S_3, \ldots$, we must find the largest of $\frac{p(n)}{n}$ for $n = 99, 999, \ldots$. For $n_d = 10^d - 1$, we have that

$$\frac{p(n_d)}{n_d} = \frac{9^d}{10^d - 1}$$

is strictly decreasing in d, since the numerator increases by a factor of 9 and the denominator increases by a factor of > 10 for each unit increment of d. It follows that the maximum of $\frac{p(n)}{n}$ over all integers $n \geq 10$ is $\frac{9 \cdot 9}{99} = \boxed{\dfrac{9}{11}}$.

Problem 3. *[6]* Let $F(x_1,\dots,x_n)$ be a polynomial with real coefficients in $n>1$ "indeterminate" variables $x_1,\dots,x_n$. We say that F is *n-alternating* if for all integers $1\le i<j\le n$,

$$F(x_1,\dots,x_i,\dots,x_j,\dots,x_n)=-F(x_1,\dots,x_j,\dots,x_i,\dots,x_n),$$

i.e. swapping the order of indeterminates x_i,x_j flips the sign of the polynomial. For example, $x_1^2x_2-x_2^2x_1$ is 2-alternating, whereas $x_1x_2x_3+2x_2x_3$ is not 3-alternating.

Note: two polynomials $P(x_1,\dots,x_n)$ and $Q(x_1,\dots,x_n)$ are considered equal if and only if each monomial constituent $\alpha x_1^{k_1}\dots x_n^{k_n}$ of P appears in Q with the same coefficient α, and vice versa. This is equivalent to saying that $P(x_1,\dots,x_n)=0$ if and only if every possible monomial constituent of P has coefficient 0.

(1) *[2]* Compute a 3-alternating polynomial of degree 3.

(2) *[4]* Prove that the degree of any nonzero n-alternating polynomial is at least $\binom{n}{2}$.

Solution:

(1) Consider the degree 3 polynomial $x_1^2x_2+x_2^2x_3+x_3^2x_1-x_1x_2^2-x_2x_3^2-x_3x_1^2$. It is 3-alternating, because, for instance, the swap of x_1,x_2 yields the polynomial $x_1x_2^2+x_2x_3^2+x_3x_1^2-x_1^2x_2-x_2^2x_3-x_3^2x_1$. This polynomial is cyclic in x_1,x_2,x_3, so the swaps of x_2,x_3 and x_3,x_1 act similarly.

(2) Let $\mathscr{C}$ be the collection of all monomials of the form $x_1^{k_1}\dots x_n^{k_n}$, where $k_1,\dots,k_n$ are non-negative integers. For any permutation $\sigma:\{1,\dots,n\}\to\{1,\dots,n\}$, there is a natural action $\mathscr{C}\to\mathscr{C}$ via $x_1^{k_1}\dots x_n^{k_n}\mapsto x_{\sigma(1)}^{k_1}\dots x_{\sigma(n)}^{k_n}$. Since σ is a bijection, the natural action associated to σ is also bijective.

Suppose that for some monomial $\mathfrak{M}=x_1^{k_1}\dots x_n^{k_n}$, there is an equality $k_i=k_j$ for some indices $1\le i<j\le n$. Then, $\mathfrak{M}$ is invariant under the action of transposition $(i\ j)$. By bijectivity, this action maps no other monomials to $\mathfrak{M}$. Hence, if the coefficient of $\mathfrak{M}$ in F is nonzero, then the coefficient of $\mathfrak{M}$ in

$$F(x_1,\dots,x_i,\dots,x_j,\dots,x_n)+F(x_1,\dots,x_j,\dots,x_i,\dots,x_n)$$

is also nonzero. Thus, if F is n-alternating, then the coefficient of $\mathfrak{M}$ is zero. Therefore, if $\mathfrak{M}=x_1^{k_1}\dots x_n^{k_n}$ is a monomial of nonzero coefficient in an n-alternating polynomial F, then it is necessary that $k_1,\dots,k_n$ are mutually distinct. It follows $k_1+\dots+k_n$ is at least $0+1+\dots+n-1=\binom{n}{2}$, completing the proof. $\square$

Problem 4. *[5]* Show that for any three positive integers a,m,n such that m divides n, there exists a positive integer $k \leq \frac{n}{m}$ such that $\gcd(a,m) = \gcd(a+km,n)$.

Solution:

It suffices to prove this statement for the case $\gcd(a,m) = 1$. To see why, we can simply divide *any* a,m,n by $\gcd(a,m)$ and obtain whole numbers $a' = \frac{a}{\gcd(a,m)}$, $m' = \frac{m}{\gcd(a,m)}$, and $n' = \frac{n}{\gcd(a,m)}$ such that $m' \mid n'$. In particular, $\frac{n}{m} = \frac{n'}{m'}$ and for integers $1 \leq k \leq \frac{n'}{m'}$ we have $\gcd(a'+km',n') = \frac{\gcd(a+km,n)}{\gcd(a,m)}$ which equals 1 iff $\gcd(a,m) = \gcd(a+km,n)$. From here we assume that $\gcd(a,m) = 1$; we aim to construct an integer k satisfying the problem condition.

Let k be the product of all primes dividing n but not a or m (we set $k = 1$ whenever there are no such primes). Notice $km \leq n$. Moreover, by construction, $\gcd(a,km) = 1$ and in fact every prime p dividing n divides exactly one of a or km. Thus, for every prime $p \mid n$, we have $p \nmid a+km$. Hence, $\gcd(a+km,n) = 1$. $\square$

Problem 5. *[7]* Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)+f(y)^2) = f(x)^2 + y^2 f(y)^3.$$

Here $\mathbb{R}$ denotes the usual real numbers.

Solution:

Let $P(x,y)$ denote the given assertion. We claim the only solution to the functional equation is $f(x) \equiv 0$ which clearly works.

First, $P(x,0) \implies f(f(x)+f(0)^2) = f(x)^2$.

Now, for all reals y,z

$$P(f(z)+f(0)^2, y) \implies f(f(z)^2+f(y)^2) = f(f(z)+f(0)^2)^2 + y^2 f(y)^3 = f(z)^4 + y^2 f(y)^3$$

By symmetry, this implies that $f(z)^4 + y^2 f(y)^3 = f(y)^4 + z^2 f(z)^3$ for all reals y,z. Taking $y=0$ gives $f(z)^4 = f(0)^4 + z^2 f(z)^3$.

Consider $z = f(w)+f(0)^2$ for some real w. Plugging this into the above gives

$$\begin{aligned} f(w)^8 = f(f(w)+f(0)^2)^4 &= f(0)^4 + (f(w)+f(0)^2)^2 f(w)^6 \\ &\implies 2f(w)^7 f(0)^2 + f(w)^6 f(0)^4 + f(0)^4 = 0. \end{aligned}$$

First, suppose $f \neq 0$ always. Then the above equation implies that $2f(w)^7 + f(w)^6 f(0)^2 + f(0)^2 = 0$ for all reals w. Therefore all the possible values $f(w)$ that f can take on are contained in set of roots of the above degree 7 polynomial.

Now as we have $f(f(x)+f(0)^2) = f(x)^2$, if some value that f takes on has magnitude > 1, we can repeatedly apply this relation to see that f attains over 7 different distinct values. The same holds if f takes on a value that has magnitude < 1. Hence $f(x) = \pm 1$ for each *individual* real x.

By taking $w=0$, we have that $2f(0)^7 + f(0)^8 + f(0)^2 = 0$. We have $f(0) = \pm 1$, and $f(0) = 1$ fails, so $f(0) = -1$. Therefore, the polynomial equation $2f(w)^7 + f(w)^6 + 1 = 0$ holds for all reals w. Again $f(w) = \pm 1$ for each individual w, yet $f(w) = 1$ does not satisfy the above polynomial equation, so $f(w) = -1$ for all reals w. The solution $f(x) \equiv -1$ clearly fails the original functional equation.

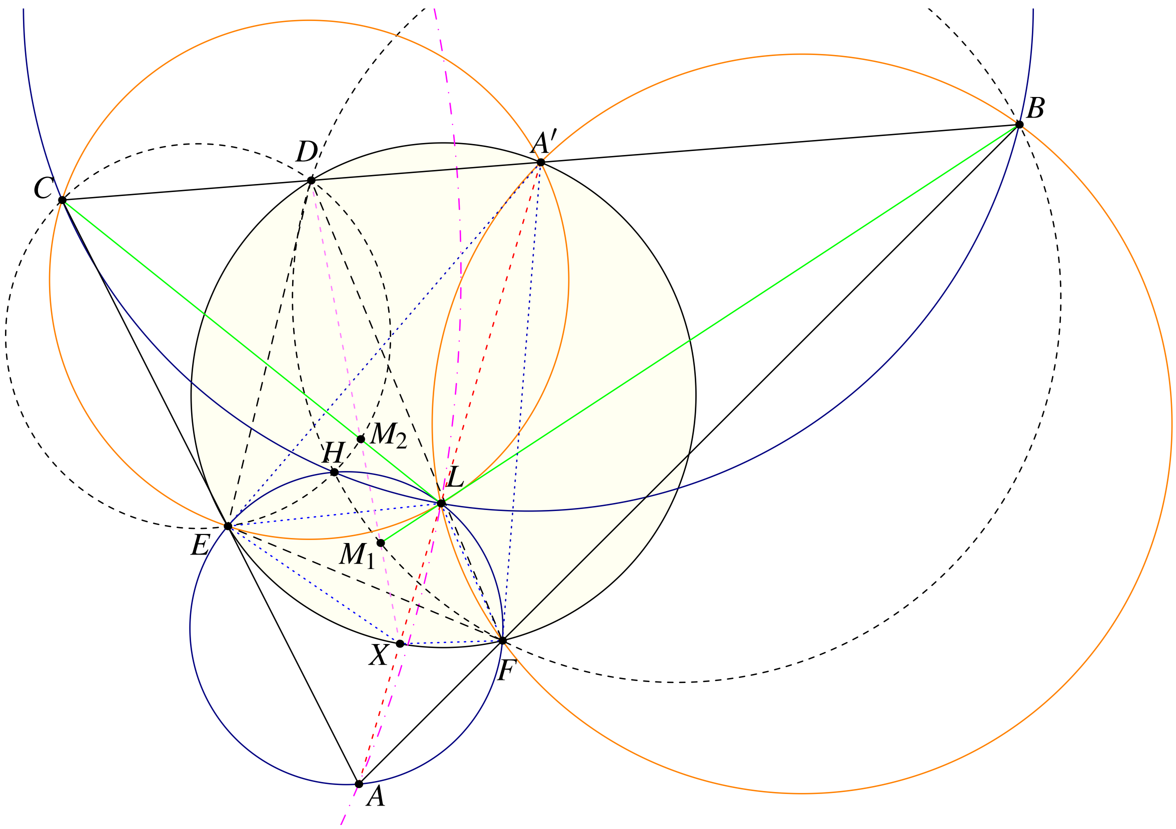
Next, suppose there exists some c with $f(c) = 0$. Taking $w = c$ above gives $f(0) = 0$. Now $f(z)^4 = z^2 f(z)^3$ means $f(z) = 0$ or z^2 for each *individual* real z. Say there is some $x \neq 0$ where $f(x) = x^2$. By $P(x,x)$, we have either 0 or $(x^2+x^4)^2$ must equal $f(x^2+x^4) = f(f(x)+f(x)^2) = f(x)^2 + x^2 f(x)^3 = x^4 + x^8$, and both cases are impossible.

In conclusion, the only solution is $f(x) \equiv 0$. $\square$

Problem 6. *[7]* Let ABC be an acute triangle with orthocenter H. A point $L \neq A$ lies on the plane of ABC such that $\overline{HL} \perp \overline{AL}$ and $LB : LC = AB : AC$. Suppose $M_1 \neq B$ lies on $\overline{BL}$ such that $\overline{HM_1} \perp \overline{BM_1}$ and $M_2 \neq C$ lies on $\overline{CL}$ such that $\overline{HM_2} \perp \overline{CM_2}$. Prove that $\overline{M_1M_2}$ bisects $\overline{AL}$.

Solution:

Denote D, E, and F the feet of the A-, B-, and C-altitudes of $\triangle ABC$; we clearly have $L \in (HEAF), M_1 \in (HFBD), M_2 \in (HDCE)$. Let A' be the midpoint of $\overline{BC}$.



Lemma 1: $L \in \overline{AA'}, (BHC)$.

Proof 1: By definition L lies on the A-Apollonius circle of $\triangle ABC$. Note

$$\frac{BE}{CF} = \frac{\sin(\angle CAB) \cdot AB}{\sin(\angle CAB) \cdot AC} = \frac{AB}{AC},$$

so the spiral center of $\overline{BE}$ and $\overline{CF}$ also lies on the A-Apollonius circle. Moreover, the spiral center of $\overline{BC}$ and $\overline{EF}$ lies on $(HEAF)$ and is not A. By properties of spiral similarities these spiral centers coincide. We deduce L is the Miquel point of self-intersecting cyclic quadrilateral $BCFE$ with circumcenter A'. The conclusion follows by well-known properties of Miquel points. ■

From this lemma we make a few key observations. By the "three-tangents lemma" $\overline{AE}$ and $\overline{AF}$ are tangent to $(HEAF)$, i.e. quadrilateral $AFLE$ is harmonic. Moreover, the homothety of center A and factor $\frac{1}{2}$ maps (BHC) to $(DEFA')$ and L to the midpoint X of $\overline{AL}$, i.e. $X \in (DEFA')$.

We finish by showing $M_1, M_2 \in \overline{DX}$. Since $AFLE$ is harmonic, $\overline{EF}$ is the E-symmedian in $\triangle LEA$, so $\measuredangle FAL = \measuredangle FEL = \measuredangle AEX$ and $\triangle AFE \sim \triangle XLE$ directly. Note A', C, E, L are concyclic by inversion of center A and radius $\sqrt{AH \cdot AD}$. Thereby,

$$\begin{aligned}\measuredangle XDA' &= \measuredangle XEA' = \measuredangle XEL + \measuredangle LEA' = \measuredangle AEF + \measuredangle LEA' \\ &= \measuredangle DEC + \measuredangle LCA' = \measuredangle DM_2C + \measuredangle M_2CD = \measuredangle M_2DB\end{aligned}$$

as needed. A similar argument shows $\measuredangle XDA' = \measuredangle M_1DB$. $\square$

Remark: L is the A-Humpty point of $\triangle ABC$; X is the A-Dumpty point of $\triangle AEF$.